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ABSTRACT

This yearbook is the final report of the National Council Committee on Arithmetic. Having endorsed the "meaning theory" of arithmetic instruction, an attempt is made to develop a position that might serve as a basis for improvement of the arithmetic program. After a chapter on the function of subject matter in relation to personality and one on curriculum problems, there follows a chapter on each of the early grades, middle grades, and high school. The next two chapters present discussions of the social phase of arithmetic instruction and enrichment activities, respectively. Chapter 9 presents a discussion of the present status of drill, followed by a chapter on evaluation. New trends in learning theory are applied to arithmetic in chapter 11. Chapter 12 poses many questions for introspective examination. The yearbook concludes with a chapter on interpretation of research followed by listings of 100 selected studies and 100 selected references. (LS)

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The National Council of Teachers of Mathematics

SIXTEENTH YEARBOOK

ARITHMETIC
IN
GENERAL EDUCATION

THE FINAL REPORT
OF THE NATIONAL COUNCIL
COMMITTEE
ON ARITHMETIC

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EDITOR'S PREFACE

This is the sixteenth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The titles of the first fifteen Yearbooks are as follows:

1. A Survey of Progress in the Past Twenty-Five Years.
2. Curriculum Problems in Teaching Mathematics.
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15. The Place of Mathematics in Secondary Education.

The present Yearbook is the final report of "The National Council Committee on Arithmetic," sponsored and financed by the National Council of Teachers of Mathematics. It should be an excellent companion volume for the Tenth Yearbook on "The Teaching of Arithmetic," which has had a very wide circulation. These two Yearbooks constitute valuable material for teachers of arithmetic and for those who have charge of teacher-training courses.

As editor of the Yearbooks, I wish to express my personal appreciation to The National Council of Teachers of Mathematics and to the members of the Committee for making this Yearbook possible.

W. D. Reeve

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ARITHMETIC
IN
GENERAL EDUCATION

Chapter I

INTRODUCTION

BY R. L. MORTON
OHIO UNIVERSITY

An examination of materials published since the beginning of the twentieth century on arithmetic as a part of the curriculum of the elementary school reveals a variety of points of view but a clearly discernible trend. Many recall a period when the doctrine of faculty psychology and a correlative belief in a large measure of transfer of training found widespread acceptance among teachers. Arithmetic became a difficult and, to many, an uninteresting subject. Failures among pupils were common; indeed, more pupils "failed" in arithmetic than in any other elementary school subject.

Emphasis upon skill in computation. The advent of standardized tests, some thirty years ago, focused attention upon the elements of computational skill. Pupils and their teachers were judged by the number of examples of a prescribed kind which they could solve in a given number of minutes with a "standard" per cent of accuracy. Drill was the prevailing mode of instruction. A few pupils who discovered for themselves something of the nature of the number system and who found meaning in the operations which they performed became both accurate and rapid in computation. Furthermore, some of them could use with ease the skills which they had acquired. They learned to do quantitative thinking. But the majority did not understand the number system and were unable to apply what arithmetic they had learned to the solutions of everyday problems. Failures continued to be numerous and there was evident a definite dislike for arithmetic.

The flight from arithmetic. Arithmetic had become an important part of the curriculum not only because it was believed to have disciplinary value but also because it was universally believed to be a subject of great practical worth. However, investigations of the extent to which men and women in various walks of life used the arithmetic skills which they had acquired revealed that they used far less than they were supposed to have learned. For the most part, too, only the simpler operations were performed. For example, a few of the more common fractions accounted for nearly all the ordinary uses of fractions in life activities.

The result was a considerable reduction in the number and variety of examples which children were asked to solve. The program became easier and, presumably, of greater "practical" worth. Unfortunately, however, this reduced program failed to develop a widespread intelligence in the use of number. In out-of-school and after-school activities arithmetic did not seem to be applied with any greater degree of success than before. It became apparent that the problem could not be solved merely by reducing the program, although the elimination of unreal and irrelevant material doubtless constituted a forward step.

The Committee's position. Soon after it was organized, the National Council Committee on Arithmetic (as the Committee responsible for this Yearbook was named) published in three journals a statement of basic points of view to which it subscribed.[1] This statement denied the validity of both the drill theory and the incidental-learning theory of arithmetic instruction and endorsed what had been called the "meaning theory." The statement set forth a series of nineteen pronouncements with reference to arithmetic as a phase of the child's school and out-of-school experience and classified these under four general headings, namely: (1) selection of content, (2) organization and grade placement, (3) methods of teaching, and (4) measurement and evaluation. Readers of the journals in which this statement appeared were invited to submit criticisms of the Committee's position. No adverse comments were received. In offering this statement and in developing a Yearbook based thereon, the

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[1] *Curriculum Journal*, 9 : 299-305, November, 1938; *The Mathematics Teacher*, 31 : 267-272, October, 1938; *The Journal of the National Education Association*, 27 : 205-206, October, 1938.

Committee is desirous of reaching a position upon which the teaching profession may agree and which may serve as a basis for a unified effort to improve the arithmetic program.

The preparation of this Yearbook. The Tenth Yearbook of the National Council of Teachers of Mathematics was devoted to arithmetic. That Yearbook found ready acceptance with the educational public. The first printing was sold in less than two years and another printing was ordered. The present Yearbook differs from the Tenth Yearbook in several important respects. It is the work of a Committee, with the assistance of a few additional contributors, rather than the work of scattered individuals working alone. The Committee *planned* this Yearbook and assigned subjects for chapters in an effort to include the more important topics and to give the Yearbook unity and coherence. Furthermore, each chapter was submitted to every member of the Committee and was then revised and rewritten in the light of the criticisms received. Thus, each chapter, instead of being the brain child of one person, may be said to be the joint product of several persons.

Nevertheless, the point of view expressed in a chapter is primarily the point of view of the author of the chapter. Naturally, each author reserved the right to accept or reject the suggestions which he received. Hence, each author accepts the responsibility for the statements which his chapter contains. Also, if a chapter makes a worthy contribution, the credit goes to the author.

This means that the different chapters express somewhat different points of view and give different emphases. The Committee believes that these differences are neither great nor serious. The Committee does not decry the differences which exist. It believes on the other hand, that teachers and other students should consider these various points of view and undertake to evaluate them. For example, the reader who turns to Dr. Brueckner's chapter after having read Dr. Wheat's chapter will gain a somewhat different idea of the nature and function of arithmetic as a phase of elementary school experience. Surely each reader will be interested in and stimulated by both of these contributions.

Those most familiar with the modern progressive movement in American elementary education may conclude, after reading the Yearbook, that insufficient attention has been given to the opportunities in a Progressive program for teaching arithmetic. The Committee has been disappointed in the fact that two persons, both of whom have been rather actively identified with the Progressive movement, were unable to make the contributions which they had intended to make. Dr. C. L. Cushman, then in the Denver Public Schools and now at the University of Chicago, was a member of the Committee for about a year but felt obliged to resign because of the press of other responsibilities. Dr. Paul R. Hanna, a member of the Committee, fully intended to prepare a chapter on the topic, "Arithmetic and the Integrated Program," but within two weeks of the date for submitting final drafts of manuscripts, was obliged to write that he would be unable to do so. The Committee deeply regrets its failure to obtain contributions from these two sources.

The topics treated. The order in which the subsequent chapters have been arranged is not a chance one but was deliberately chosen with due consideration of the expressed preferences of all the members of the Committee. It is believed to be a logical one. However, no serious loss will be experienced by the reader who chooses to read the chapters in an order different from that in which they have been presented.

Dr. Buswell's chapter on "The Function of Subject Matter in Relation to Personality" suggests a value for arithmetic which will not have occurred to many readers. It is a unique contribution and, in the opinion of other members of the Committee, a significant one. Dr. Sueltz presents a discussion of curriculum problems and the complex and controversial issues having to do with grade placement. His discussion is timely since there is a widespread interest in these topics. It will be particularly valuable to those engaged in curriculum revision and reorganization.

The chapters by Dr. Thiele, Dr. Wheat, and Dr. Benz stress aspects of arithmetic teaching which will be of special interest to teachers in the primary grades, the intermediate grades, and the high school, respectively. Dr. Thiele's extensive experimental work with pupils in the lower grades at Detroit makes him exceptionally well qualified to discuss the role of generalization in the learning of arithmetic fundamentals. Dr. Wheat's well-known contributions to the psychology of the elementary school subjects, particularly arithmetic, will guarantee special interest in his chapter in which a theory of instruction is developed. Dr. Benz, who has had extensive experience in the theory and practice of secondary education, presents in a challenging manner the contribution which arithmetic may make to the education of secondary school pupils.

The Committee has frequently stressed what seem to be two outstanding phases of arithmetic, viz., the *mathematical* phase and the *social* phase. This does not mean that the Committee conceives of arithmetic as functioning separately and independently in the lives of children as mathematics on the one hand or as a social experience on the other. To insure that the reader will find that the social phase is given proper emphasis, Dr. Brueckner has made it the subject of his chapter. The contribution which the pupil's experiences may be expected to make toward learning arithmetic as a "system of ideas" is also recognized by Dr. Wheat.

The Committee believed that to be of maximum usefulness to teachers and supervisors the Yearbook should contain a collection of materials and devices which others have found most helpful in enriching the course. Miss Sauble's position at Detroit has given her the experiences which have qualified her to prepare such a chapter. The Committee requested her to assemble this material and she has done it admirably.

Now that the drill theory of arithmetic instruction is in disrepute, there is often expressed concern as to what is to become of drill. At the Committee's request, Dr. Buckingham has discussed drill as a phase of the new type of arithmetic program and has shown that the term "drill" may be invested with a richer meaning than has been associated with it in the past.

Likewise, there is deep concern these days over ways and means of evaluating learning, inasmuch as many if not most of the tests formerly in use were designed to measure the results of learning efforts which were based upon the drill theory. If the "meaning theory" is to prevail, how shall learning be evaluated? Dr. Brownell has discussed this subject at length and has provided concrete suggestions for putting into effect a newer type of evaluation program.

In Dr. McConnell's chapter, the reader will find a skillfully organized statement on the subject of recent trends in learning theory. For many, it will throw new light on the psychology of arithmetic. This chapter may be more difficult reading than other chapters for those who have done little reading on the psychology of learning in recent years, but the mere fact that the reader finds it difficult is in itself evidence that it will be worthwhile to read and reread the chapter until the implications of recent trends in the theory of learning are fully appreciated.

A list of questions. The Committee believes that the teacher will profit from an occasional introspective examination of his own qualifications. To facilitate such a self-examination, Dr. Wren has prepared a list of questions and has organized these questions in two major divisions and ten sub-classes. Many readers will find that these questions point the way toward further professional preparation.

In recent years, arithmetic has been the subject of many research projects. The interpretation of the results of research is often difficult, partly because these results sometimes are at variance with what ordinary observation and common sense seem to indicate, and partly because the results of one study may disagree with those of another. Dr. Brownell and Dr. Grossnickle, both of whom are well known for their own contributions to research in arithmetic, have written a chapter which the reader will find a valuable aid in interpreting research.

The bibliographies. So many books, monographs, and articles on arithmetic have been published that the teacher cannot expect to read more than a minor fraction of the total. In attacking the difficult task of selecting a bibliography, Dr. Stretch and Dr. Bond assembled many hundreds of references and, with the help of other members of the Committee, undertook to choose a list of usable length. Dr. Stretch finally brought together one hundred selected research studies, and Dr. Bond one hundred selected references which are largely non-quantitative in character. These lists are presented in the last two chapters with the hope they will be helpful to teachers who wish to do further reading and study.

Obviously, it is impossible in a single volume to consider all the issues which have to do with arithmetic as a part of the school program. The Committee has undertaken to provide a discussion of those which seem to be the most important. This Yearbook is offered with the hope that it will assist the teacher in planning a better program and in providing for pupils richer and more meaningful arithmetic experiences.

Chapter II

THE FUNCTION OF SUBJECT MATTER IN RELATION TO PERSONALITY

BY GUY T. BUSWELL
UNIVERSITY OF CHICAGO

The teaching of arithmetic has commonly been evaluated in terms of the development of mathematical skills, the ability to think quantitatively, and the ability to apply arithmetic in social situations. Few would deny that these are significant ways of measuring the value of an arithmetic program. On the other hand, the outcomes of arithmetic may also be evaluated in terms of their contribution to the development of personality. This latter basis for evaluating arithmetic has become prominent in the literature only during the last decade, although, in essence, it is not a new criterion since it is only a particular illustration of one of the aspects of transfer of training. Furthermore, not only arithmetic but all subject matter is equally open to examination in terms of its contribution to the development of a child's personality.

Positive and negative views. One does not need to search far in the educational periodicals to find statements, both positive and negative, relative to the effect of subject matter on personality. For example, in *Childhood Education* Nina Jacobs[1] defends the proposal that arithmetic has important and positive values for the development of personality. She gives numerous concrete illustrations from her own classwork to show how an understanding of number contributes to the development of a sense of security, of the idea of responsibility, and of the feeling of necessity for cooperating with others. She defends the position that no subject gives a greater sense of security than does mathematics.

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[1] Nina Jacobs, "Personality Development and Adjustment through Number," *Childhood Education*, 13 : 258-261, February, 1937.

Quite in contrast with the position defended by Miss Jacobs is that expressed in an article by Professor Lane[2] published in a later issue of the same journal. Professor Lane blames the "three R's" for much of the personality maladjustment found in young children. He takes the position that "It seems absolutely essential that reading, writing, and arithmetic shall cease to occupy the center of the attention of primary teachers. These skills are learned through a relatively small number of flashes of insight rather than through careful learning of all the elemental processes involved in them. The good teacher of the skills is one who can help children live well and richly regardless of skill equipment and who is able to detect in individual children the need for assistance in attaining insight. The good teacher never induces labor leading to the delivery of ideas or insight. . . . To primary teachers I would suggest a few marked changes in procedure: De-emphasize the three R's. From the remotest corner of the subconscious drive out the concept of respectability as related to achievement in the three R's. Make certain that every child is experiencing worthwhile attitudes and meanings regardless of his skills. Under no circumstances cause a child to lose caste or status, nor to gain it, with you or with his associates, through the skills."

It is obvious from the two preceding illustrations that in this same subject of arithmetic Miss Jacobs sees positive values which can make a marked contribution to personality whereas Professor Lane so fears negative outcomes that he would like to see arithmetic subordinated and "de-emphasized" in the school's program. Other illustrations could be cited showing such opposing points of view related to the arithmetic of the intermediate and upper grades.

Apparently, here is an issue of major importance since the nature of the arithmetic program will depend very much upon which of these two positions is taken. Furthermore, as has been said, the issue is far larger than simply the subject of arithmetic since, in essence, the problem is the extent to which organized subject matter contributes to or inhibits the development of personality. There is also involved the question whether personality is a direct or an indirect outcome of education and whether the concept of personality is adequate to cover all the desirable objectives of education.

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[2] Howard A. Lane, "Child Development and the Three R's," *Childhood Education,* 16 : 101-104, November, 1939.

The assertion that arithmetic as taught during the past twenty years produces some undesirable outcomes is not open to dispute. Certainly, the application of the drill theory has produced a meaningless outcome which no one can adequately defend. Furthermore, the attempts to introduce the subject by teaching abstract number combinations, without first building up a background of concrete understandings, has separated the arithmetic of the schools from the vivid and genuine number experiences which the child builds up in his out-of-school quantitative experiences. In the middle and upper grades the situation has been little better as the work of the pupils is all too often characterized by a formal manipulation of arithmetical processes devoid of either mathematical meaning or social significance. The need for reform in arithmetic is freely admitted. The question is: What shall be the nature of this reform?

Proposals for reform. Those who have attacked arithmetic from the standpoint of its bad effects on personality development have made three proposals. The first of these, made by only a small minority of the group, is that there be a complete abandonment of systematic school subjects, a proposal which affects not only arithmetic but all organized subject matter. No one has yet worked out a complete program to substitute for organized subject matter and the proposal is often expressed in such loose terms as Professor Lane has used when he urges teachers to "study childhood, have faith in children, provide a wealth of worthwhile child-like experiences and watch 'em grow!" This trustful faith in "watching 'em grow" (like Topsy!) is worthy of serious consideration only as it is modified by the phrase, "provide a wealth of worthwhile child-like experiences." The whole issue hinges on what should be the nature of these "worthwhile experiences." If they are to be nothing more than the incidental and extemporaneous experiences proposed by teachers and pupils from day to day there is little to look forward to except chaos. On the other hand, if these "worthwhile experiences" are the product of intelligent and serious study they are bound to eventuate in some kind of organized content which will be available for all teachers. This will still be some sort of organized subject matter and it is quite immaterial whether it is called "arithmetic," or "a school subject," or whether it goes by such names as "activities" or "projects." Abandonment of organized subject matter cannot be defended.

A second proposal for the reform of arithmetic is that the subject be postponed or that some of the content be deferred until children are more mature. There is a great deal of sense to this proposal, provided the school does not simply postpone the subject as it is now taught but rather makes a careful study of the relationship between maturity of children at different ages and the arithmetical concepts and experiences which can be made meaningful to them. The various proposals which have been made for deferring arithmetic have fallen considerably short of this criterion. For example, the widely known proposal of Benezet as described in the N. E. A. *Journal*[3] was a postponement in name only. Benezet did not postpone arithmetic to the seventh grade as a reader might infer from his statement, "If I had my way, I would omit arithmetic from the first six grades." Rather he eliminated many of the formal aspects of the subject and substituted in their place some very desirable learning experiences which simply made a different kind of arithmetic.

Certainly, few teachers are so naïve as to believe that number experiences can be postponed completely until the seventh grade or, for that matter, until the third grade. Consequently, the proposal to postpone arithmetic until children are more mature is very misleading. What the proposal really calls for is an abandonment of *formal* teaching, with which anyone will agree, and the rearrangement of number experiences to fit the developing maturity of a child, to which, also, no one can take exception. The shallow argument that pupils can learn more arithmetic in the same amount of time if the subject is postponed scarcely needs consideration since, by the same argument, everything could equally well be postponed. It is difficult to name any type of learning which cannot be accomplished more readily by an eighteen-year-old person than by one six or nine years of age, but the implication is not that all education be postponed and presented as a lump as soon as maturity is reached.

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[3] L. P. Benezet, "The Story of an Experiment," *Journal of the National Education Association*, 24 : 241-244, November, 1935.

A third proposal of those who admit the deficiencies of arithmetic as it has been taught, and who are interested in much more than just a school subject and are, consequently, concerned with the effect of arithmetic on personality, is that a major reorganization of subject matter and methods is needed, but that the outcome must be an organized body of number experiences from which both mathematical insight and social significance may be derived. Furthermore, they maintain that in this type of number experience positive contributions to the development of desirable personality traits can be found. Before analyzing this proposal a brief consideration should be given to the nature of personality development.

The nature of personality development. A personality characteristic is always a generalized rather than a specific form of behavior. For example, being courteous is not a matter of stereotyping specific responses, such as saying "Thank you" or "Excuse me," but is, rather, a generalized form of response which expresses itself in various forms of speech and behavior, and always arises from the recognition of a set of values relating to other people and reflecting attitudes of respect and consideration for them. Personality characteristics grow out of experiences.

A commonly accepted hypothesis holds that personality develops through adjustment to frustrations in experience. The way a person adjusts to being thwarted in obtaining the things which he craves either by nature or by habit makes his personality what it is. If frustrations become too numerous or too severe the individual may develop a feeling of inferiority and a defeatist attitude. If life becomes full of fear the individual may adjust by cringing and hiding or by adopting bullying attitudes to conceal his fear. If life becomes altogether too easy and the child is able to satisfy his cravings with no effort on his own part, there is no basis for developing strength of personality and the individual takes on attitudes of smugness and arrogance which are as socially undesirable as are some of the attitudes which develop from excessive frustrations at the other end of the scale. Furthermore, experiences which lead one to see new values, to discover exactness and order in matters which once seemed confusing or which had not been in the realm of one's experience at all, are successes which add positively to personality. A child's personality emerges from his adjustments to the experiences of home and school.

Schools of the past have given altogether too little thought to the nature of this development of personality, and an overbalanced emphasis on subject matter has been the natural outcome of a theory of education which was sometimes expressed by the phrase, "Knowledge is power." An overbalance of emphasis on knowledge as the sole aim of education and subject matter as its sole means of attainment may, without question, produce maladjusted personalities. On the other hand, personality does not develop in a vacuum. Furthermore, if the organized learning experiences of the school must give way completely to unsystematic, life-like situations, then the school must face the possibility of finding itself outmoded and must be prepared to accept the proposal that life experiences are superior to those of the school and, consequently, our schools are unnecessary. But lack of schooling results in illiteracy, and illiteracy has such marked and well-known effects on personality that no one would pretend to defend it as an alternative. What the school must find is some mode of developing personality and at the same time of retaining some of the other necessary objectives of education, such, for example, as fund of understandings, of knowledge, and of skills required for complete living in such a world as this. If, as stated in our hypothesis, personality is the product of one's habitual method of meeting frustrations and resolving conflicts, how is arithmetic related in any way to such personality development? Or, stated more generally, how can education of any kind contribute to the development of personality?

The development of personality depends upon an individual's modes of control of situations which produce frustration and conflict. If one is helpless in the face of such situations, there is no alternative to feelings of defeat, inferiority, and lack of self reliance; whereas, given a successful method of meeting frustrations, the individual develops feelings of poise, assurance, and confidence. Education can contribute to these modes of control.

Mathematics and quantitative relationships. One of the large areas of human experience has to do with quantity and with quantitative relationships. It is the function of mathematics to contribute to one's ability to meet situations of this kind. Children who cannot read the clock are frustrated in making adjustments to social practices. Once they learn to read the clock they can avoid many frustrations by arranging their activities in accordance with the necessities of a time schedule. Children who have not learned to count are excluded from participation in many activities in which other children find much pleasure. Number plays a large part in the activities of even primary grade children. Because of the very basic and permeating character of mathematics it becomes a mode of control of increasing importance as children mature and the frustrations requiring control are as numerous in their thought experiences as in their overt behavior.

The advantages mentioned in the preceding paragraph are seldom pointed out when arithmetic is criticized on the ground that it damages personality. When this criticism is made it is asserted that arithmetic is dull, that it is abstract at too early a level, that it destroys interest, that it produces an aversion for school, and that it sets up habits of intellectual formality instead of vigorous mental responses. There is no doubt that all of these faults may be witnessed in some arithmetic classes. There is also no doubt that they may be found in some classes in any subject. The cause of these undesirable modes of response cannot be laid to arithmetic or to any other particular subject; they are the result of poor selection and organization of content, poor methods of teaching, and poor personality of the teacher. Formality is an attitude of mind; it has no place in any proper scheme of education. Arithmetic, or any other subject, may be taught formally and the outcomes may be damaging or undesirable. On the other hand, arithmetic, or any other subject, may be taught with complete lack of formality when content and method are so planned that understandings are vivid throughout and so that the later abstractions are always the products of earlier concrete experiences.

In considering the relationship between arithmetic and the development of personality, some specific charges should be considered.

Faults attributed to arithmetic. One of the chief objections to arithmetic has been that it is too difficult, that the percentage of failures in the subject is too high, and that because of its difficulty it causes children to dislike school in general. It is true that arithmetic is more difficult than some other more concrete forms of experience. In the very nature of the case arithmetic deals with abstract symbols, and dealing with abstractions is always more difficult than dealing with concrete experiences. On the other hand, no thinking person would deny that the special values of arithmetic reside in the fact that it does deal with abstract relationships. Quantitative relationships can be handled effectively by using arithmetical processes simply because so handling them makes it possible to abstract the essential quantitative relationships from the great mass of concrete accompaniments and to save much time, effort, and confusion in arriving at the solution. Any good education leads toward abstract concepts no matter what is the field of activity.

It is, of course, more difficult to learn to divide by a fraction than to work out the same problem concretely through the use of sticks or counters, but the reason we develop abstract processes from concrete experiences is simply because, once they are learned, the abstractions furnish a much more economical method of dealing with experience. However, there is no denying the fact that some difficulty is involved in learning any abstraction. Nevertheless, to eliminate anything that is difficult on the ground of such statements as that quoted from Professor Lane, namely, "The good teacher never induces labor leading to the delivery of ideas or insight," is open to direct challenge, and to challenge on the very grounds that a complete elimination of labor from the process of learning would be damaging in the extreme to personality development.

One of the commonest adjustments which must be made in all kinds of life experience is the adjustment to work. The necessity of work frustrates individuals in every conceivable kind of experience. A personality that has never learned to adjust to a work situation leaves much to be desired. When arithmetic is handled by good teachers, and there are many of them, it furnishes an exceptionally good opportunity to develop personality characteristics of persistence, industry, and concentration, because in the very nature of arithmetical operations the results of such activities are so clearly verifiable. The previous sentence is not a revival of a doctrine of formal discipline. The writer is not talking about the development of any hypothetical abilities or "faculties of the mind," but simply about the establishment of generalized habits of reacting to frustrations of difficulty in the ways described. As has been previously pointed out, personality is merely the sum of these generalized habits which are the outcome of experience.

The organization and grade placement of arithmetic may be so poorly managed that the pupil is frustrated by work assignments entirely beyond his stage of maturity; but, on the other hand, number experiences may be properly related to the developing maturity of children in such a manner that the difficulties imposed are stimulating rather than depressing, and the child may learn the exhilaration of mastering a series of meaningful experiences. It is because arithmetic has often been presented through barren drills rather than through meaningful experiences that criticisms of this type have appeared; but to throw out arithmetic because it is sometimes presented in too difficult a fashion is like "throwing out the baby with the bath."

Another criticism of arithmetic is that it often produces formal, meaningless responses from children, leaving them confused and with a feeling of dullness in the face of quantitative situations. There is little doubt that an overemphasis on speed in computation, with the attendant large amounts of drill, may produce such outcomes. This is no particular charge against arithmetic since it is doubtful whether any kind of content could have succeeded under such methods of teaching. The outcome, however, is by no means universal and there are many persons, the products of our schools, for whom number operations are clear and meaningful. Since the year 1930 there has been a most marked improvement in this particular respect and, as noted in the introductory chapter of this Yearbook, the position of this Committee is completely in accord with emphasis on meaning and understanding, rather than on speed in computation. There are ample grounds for this kind of criticism of arithmetic but certainly the alternative for the schools is not an abandonment of the teaching of the subject. Rather, it is the development of a much superior kind of teaching of arithmetic. It is the improvement of teaching, rather than the criticism of poor teaching, which should receive the attention of school people.

The position taken in this chapter is that arithmetic has decidedly positive values to offer to the school's program of personality development. It can help children adjust to those types of frustrations which defeat all too many adults. Some illustrations may serve to make this point clearer.

Some illustrations. In the writer's dealings with graduate students in education numerous cases have been encountered in which students have deliberately sidestepped worth-while problems because these problems involved some understanding of statistical concepts. They were frightened by any suggestion of the need for statistical understandings. These cases have exhibited a great range of forms of adjustment, even to some students who have paid sizable sums of money to secure assistance on operations so simple that a few hours of intelligent study would enable any individual to understand them. Yet they expressed a feeling of utter inadequacy in the face of any mathematical situation. They go through life with feelings of inferiority and dread simply because they have not learned enough mathematics to enable them to behave intelligently in quantitative situations. The damaging results of a lack of arithmetic in these cases are far more obvious than any damaging results which might accompany a serious attempt to understand the subject.

There are in this country a large number of women who, when they find it necessary to triple a recipe, measure out two-thirds of a cup of flour three times rather than hazard the mental operation of "How much is three times two-thirds?" Everyone knows persons who are afraid to display their lack of ability at adding scores in a card game and are much embarrassed by the fact that they are afraid to do so. Many otherwise intelligent citizens throw up their hands in surrender in the face of trying to understand the financial operations of even local government. Innumerable people are caught in the trap of installment buying because they do not know how to work out the simple arithmetic of the situation. Short-term loan agencies flourish on the patronage of people who never realize the exorbitant rates of interest they are paying until they are trapped by the impossibility of keeping pace with the rapidly accumulating size of their loans.

As Professor Wheat has pointed out, "One can remain so ignorant of number as to be very complacent in the face of the problems and frustrations of modern life. Many people are fairly well-integrated personalities merely because they know so little about the demands which number relations make in a complex world. These are the 'Ham-and-Eggers' and 'Thirty-Dollars-Every-Thursday' people. On a low level of thinking, they may be very well integrated; with some understanding they are less so on a higher level; and with more understanding they may become integrated on a higher level."

Thus, one can see in every direction examples of frustrations caused by a lack of genuine understanding of the arithmetic of quantitative experiences. The arithmetic of the elementary schools has been inadequate for these purposes. It must be improved, not eliminated.

Although the development of personality is an important obligation of the school, it is not an outcome which can be produced by direct attack such, for example, as offering courses in personality, or teaching a person how to be self-reliant, or self-confident, or "how to make friends." This kind of direct teaching involves the danger of producing introverted prigs. It may result in changes in personality, but changes of an extremely undesirable kind.

Personality adjustment through experience. Personality develops best through an education which gives the child experiences in meeting successfully the many kinds of frustrations which life brings to everyone. Many of these adjustments are of the immediate, personal kind—such as are encountered on the playground, at home, in life outside the school. These are important kinds of adjustments to which the school can undoubtedly give more intelligent consideration than it has generally given in the past. On the other hand, there are extremely important kinds of adjustments which can be made only by the mastery of some of the more important kinds of understandings which the race has learned through long ages of experience. These understandings make up the content of the greater part of the school's curriculum. The strategy of education is so to organize and present these essential concepts that they can be understood in the relatively short number of years that an individual can spend in school.

Schools should differ from life in presenting these important understandings in a way very much superior to that afforded by life outside the school. But these understandings that make up the content of the curriculum are not something apart and separate from the development of personality. They provide the controls of frustrations from which personality characteristics emerge. The mastery of significant content is one of the most direct ways in which one may be aided in his adjustment to frustration. Certainly, the difference between the way a well-educated person reacts to the frustrations of life as compared to the way an uneducated, illiterate person reacts to them needs no emphasis to the readers of this Yearbook. The central issue here is: "Are the kinds of experiences which have been organized into our so-called school subjects the best kinds of experiences which may be presented and is the organization and method of presentation likewise the best?"

In reference to the subject of arithmetic the writer's position is that the mastery of mathematical relationships is essential in making many adjustments which eventuate in desirable personality traits. The arithmetic which the schools have been presenting is admittedly too formal, often poorly graded, often presented by methods which can be improved. However, the remedy for this situation is not to eliminate it, but to experiment with it, to adjust it to the level of maturity of the child being taught, and to make it meaningful in the experiences of children. By so doing the school can make a valuable contribution to the development of essential elements of personality.

Chapter III

CURRICULUM PROBLEMS—GRADE PLACEMENT

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ARITHMETIC, because of its service to the individual and to society as well, continues to occupy a significant position in the curriculum of the elementary school. In general, the curricular problems pertaining to arithmetic are similar to those of a decade and a generation ago. They originate with the recognition that arithmetic is important in the lives of intelligent citizens and with the observation that many of our recent graduates from the public schools are not competent in simple mathematical situations.

GENERAL FACTORS IN ARITHMETIC CURRICULA

During the past generation school curricula have become crowded. Schools have experimented with "activity curricula" and with "experience curricula" in an attempt to ease the situation and to make education more meaningful and pleasurable for the child. Since experimental curricula were found readily adaptable to the social studies and to certain phases of the language arts, these areas were frequently emphasized. The net result has been that arithmetic has suffered. Not only has arithmetic suffered in the amount of time devoted to it but also, and much more to be deplored, it has suffered in the mode of instruction and in the appraisal of its nature and function. Too commonly, teachers and school officers have viewed arithmetic as a group of skills which are to be memorized so that having been learned they are useful tools which are always readily available. Several chapters of this Yearbook present a sounder and more rational view of arithmetic and of better methods of teaching and learning.

This chapter will discuss such important questions as arise when a curriculum committee faces the tasks of selection, arrangement, and placement of arithmetic materials in a curriculum for the elementary school. In the main, the discussion will be based upon the implications of the position adopted by the Yearbook Committee. Naturally, however, the personal opinion of the writer cannot be wholly excluded. In order that the reader may better understand later arguments, the breadth of aims in a modern program of arithmetic will be noted.

Types of aims in arithmetic. Curricular problems as well as problems dealing with instruction and evaluation should be viewed in terms of the educational aims which are expected. The school that expects to achieve only temporary mastery of abstract computational problems faces a task different from that of the school that has a much broader vision of the nature and function of arithmetic.

Many different arrangements and descriptions of the aims of arithmetic are possible. The following classification of types and kinds of aims shows briefly the scope of a modern program.

1. *Concepts and Vocabulary.* This includes all the vocabulary items ranging alphabetically from *add* to *zero,* together with the ideas and concepts associated with these language elements. Concepts, as for example those of measures and fractions, should be rich in associations so that a child may gain ideas and visualizations of size or value and of use.

2. *Principles and Relationships.* This type of aim includes such principles as the mathematical equality, for example, of ten nickels and two quarters, the principle of place value in writing numbers, and all such relationships as the comparisons of numbers and measures.

3. *Social and Economic Information.* In this class are found the many items of consumer information, such as "coffee is sold by the pound," the interpretation of statements such as "an 80 ft. lot," and such business forms as receipts and checks.

4. *Factual Information and Materials.* This includes such items as the identification of geometric forms, the common facts and equalities of measures, and the simple number facts or combinations. This class is distinguished from the preceding one in that

here the information is more definitely in the realm of materials of mathematics, while in the former the information is more in the realm of applications of arithmetic.

5. *Processes and Manipulations.* This type comprises all the computations ranging from simple addition through percentage and including work with fractions, with decimals, and with standard measures.

6. *Problems and Basic Thought Patterns.* This group of aims includes the early associations of processes with arithmetical situations, e.g., addition becomes associated with the situation of combining quantities. Common basic types of problems such as those associated with *cost, number,* and *price* and with *distance, rate,* and *time,* as well as more specialized types of problems such as those occurring in percentage are included.

7. *Reflections and Judgments.* Many types of reflection which are not always recognized as mathematical are here included. For example, extracting pertinent data from a situation or description, judging whether another's reasoning is correct, drawing inferences from data, comparing similar elements in different expressions, and seeing the relationship of two variables often require mathematical abilities not included in the usual arithmetic program.

It is easily apparent that the above groups of aims are closely interrelated and that attempts to teach any one type will usually foster and extend learning of several other types. Again, each of the above groups is important because it contributes to intelligent participation in social, economic, and cultural affairs. A program or curriculum in arithmetic may be judged in terms of the extent to which it features all the broad aims in such a way that pupils will become able to think and to act intelligently in arithmetical situations.

The aims and expected outcomes of a program of arithmetic should be divided and grouped in terms of the type of learning or the degree of mastery expected from the pupils. Three types or levels of learning may be anticipated by the teacher:

1. *For Permanent Mastery*—the concepts, ideas, items of information, principles, habits and patterns of thinking, and the computations which all of us need in the conduct of our affairs, e.g., subtraction, comparisons, measurements, etc.

2. *For Temporary Understanding*—a considerable amount of arithmetic which is usually taught and frequently forgotten by the pupils but which may be readily relearned and is available for reference, e.g., lesser used facts of measurement and semi-technical computations.

3. *For Partial Understanding and Appreciation*—mathematical ideas, principles, information, and processes which the ordinary citizen should know exist and which he should partially understand but the mastery of which is reserved for specialized workers, e.g., foreign currency, "case III" percentage, and finding the rate in an installment-buying problem.

To be sure, curriculum workers will disagree on the content for each of the three groups or types of learning to be expected. Some committees may wish to go much further and designate levels of learning for different groups of pupils. However, having recognized a problem, a curriculum committee should seek an answer. The solution involves such factors as the pattern of education which is dominant in the school, the committee's experience with and insight into arithmetic, the committee's knowledge of the learning activities of boys and girls, and similar elements which affect any phase of learning. During the next decade much progress should be made in the reclassification of aims of arithmetic in terms of the types and kinds of learning to be expected. Too frequently teachers attempt to teach for permanent mastery all the materials in a textbook. Many textbooks have already adopted plans, such as starred items or sectioned pages, in order to provide different types of materials for different pupil abilities.

Trends in elementary education. Obviously, the organization and the pattern of the elementary school together with the methods of teaching and learning employed therein affect the curriculum. They influence the selection, the organization, and placement of materials as well as the effectiveness of learning. In order to show how the type of school affects the learning of arithmetic, three generalized types of school organization will be described briefly.

1. *The Traditional School.* The traditional school is divided into grades in which children study such subjects as reading,

arithmetic, science, and literature. These school subjects are divided into sections so that in each school grade a pupil is expected to learn certain definite things. The pupil is promoted from one grade to another in terms of his achievement in the school subjects. The curriculum of this traditional school is organized according to the school subjects.

2. *The "Activity-Experience" School.* While individual schools of this type differ markedly, they are similar in that school subjects are of importance only as they contribute to the larger activities and experiences in which the pupils engage. Typically the pupils of a fourth year might be studying transportation or communication and while doing this they learn such reading and arithmetic as are encountered in the investigation of the topic of transportation or communication. Subject matter as conceived in the traditional school is secondary. In the "activity-experience" school the curriculum is usually expressed in terms of experiences rather than achievement in subject fields.

3. *The "Combination" or "Fusion" School.* Many schools are attempting to combine or fuse the better elements of the traditional school with those of the "activity-experience" school. This is frequently done by organizing the curriculum in terms of subject-matter sequences and goals and adopting "activities" and "experiences" when these contribute toward the achievement of the subject-matter goals. Occasionally the lower grades of an elementary school follow the "activity-experience" pattern, while in the intermediate and upper grades emphasis is shifted gradually toward a more traditional pattern.

The large majority of elementary schools in the United States follow the traditional pattern. A growing minority are of the "combination" or "fusion" type. The general trend is toward humanizing education and making it more meaningful to the pupil. In arithmetic this means a reappraisal of both content and method in terms of their social, economic, and cultural significance.

In the more common traditional school organization, the arithmetic curriculum is built around the logical sequences and mathematical dependencies which characterize the processes of arithmetic. Because these logical relationships are so evident to one who understands arithmetic, they frequently lead a teacher to regard the subject as an abstract science of numbers. As a result teaching may become stereotyped and the aims may be narrowed much more than is desirable. The mathematical relationships of arithmetic aid greatly in the development of meaning in processes and computations, but they alone will not ensure that pupils grasp the social and economic significance of their learning.

In the main, investigations of curriculum problems during the past twenty-five years have dealt with arithmetic in the traditional school. They have dealt with selection, organization, and grade placement of subject matter, and this has usually been a narrow interpretation of subject matter. Curriculum makers are still puzzled over the problem of giving breadth, meaning, and significance to arithmetic teaching. A curriculum may suggest that teachers capitalize upon and enlarge the mathematical experiences of their pupils in order to help them to sense and to resolve mathematical situations in the real world, but it cannot guarantee that these things will be done. That is a responsibility that must be shared by teacher education and by school administration.

In "activity-experience" schools the learning of arithmetic depends in large measure upon the wisdom and judgment of the teacher. Arithmetic becomes a secondary aim and frequently a very casual one. Two weaknesses of this type of school are apparent: first, pupils and teacher alike are likely to miss an important mathematical situation because they are unfamiliar with it and its implications; and second, pupils and teacher are likely to investigate a situation or problem and attempt to master it without regard to its place in a sensible mathematical sequence. Furthermore, the investigation and study of a topic such as transportation in all its aspects does not provide for the intensive study that is needed to learn to master a mathematical process.

The "activity-experience" curriculum may serve very well for the early stages of the development of concepts and for giving social and economic significance to much of arithmetic. However, if fuller comprehension and genuine usefulness of processes is to be achieved, a definite program of instruction in arithmetic is needed. The issue reverts to the aims of the school. In "activity-experience" schools, if arithmetic is to be learned, the teachers must know when the mathematical element in an "activity" should be assumed or furnished and when it should be thoroughly investigated and studied. Teachers in these schools should possess a broad knowledge of the nature and uses of arithmetic.

In schools that follow a somewhat traditional pattern but also draw freely upon the values of pupil's activities and experiences, a fine type of arithmetic learning may be achieved. This is possible when a good curriculum is being carried out by a well-educated teacher who knows how and when to use the logical relationships of arithmetic and when social experiences should assume dominance. In all types of schools there are many different ways of approaching and achieving learning. Some teachers excel in one approach while others prefer another. One method may appeal to certain children and not to others.

The organization of a school should be sufficiently flexible so that teachers may capitalize upon their own particularly suitable methods of teaching. Similarly, curricula for various subject fields should allow for differences of treatment and modes of learning. Procedures may become quite as stereotyped in an "activity-experience" school as in any traditional school. Both types offer advantages. The future will probably witness a fusion of the two in such a way that each may best contribute toward the education of boys and girls.

QUESTIONS CONFRONTING SCHOOLS

Despite divergence in organization of schools and in philosophy of education, certain questions concerning the nature and place of arithmetic in the whole curriculum arise repeatedly. Most of these problems and questions cannot be answered categorically. At best, any discussion of questions pertaining to the curriculum reflects the ideals and experiences of the writer.

The more commonly asked questions that deal with arithmetic in the school curriculum will be stated and discussed in three groups: (1) those dealing primarily with the selection of content, (2) those dealing with arrangement and placement of materials, and (3) those dealing principally with instructional problems.

Inasmuch as several chapters of the Yearbook deal with instructional problems, the discussion here will be very brief. The general aim is to direct thinking, rather than to offer specific solutions.

Selecting the arithmetic content. In selecting the arithmetic content several questions may be asked:

1. *Who should determine the nature and scope of the arithmetic curriculum? What part should the pupils, their parents, the teachers, school officers, and curriculum specialists share in this job?*

Perhaps the best answer here is that those who understand most about the job should be entrusted with it. The amount of arithmetic or of any field of study in the whole curriculum is dependent upon its importance in the lives of people. Hence, those who make the decision ought to be qualified to sense and to interpret the mathematical elements in our social, economic, and cultural activities. Building a curriculum in arithmetic is like building a bridge. A specialist may be called in to design a bridge but his utility is slight unless he understands modes of construction and strengths and values of materials. In general it is perhaps best to build a curriculum through the cooperative efforts of school officers and teachers who understand both children and arithmetic. To date, the problems concerning curriculum making are largely in the realm of personal opinion. It should be an *informed* opinion.

2. *Is there too much arithmetic in the elementary school?*

This question has several aspects. Perhaps some of the arithmetic now commonly taught in the elementary school should be shifted to the high school. Such a suggestion is made in Dr. Benz' chapter of the Yearbook. Certain phases of arithmetic, such as the uses of percentage and computations of volumes and capacity, have greater utility for adults than for children. Certain processes, such as some calculations of interest and the indirect cases of areas and volumes, are fairly difficult for elementary school children but not for high school children. Many teachers feel it would be highly desirable to lessen the arithmetic load in the elementary school by transferring some of it to the high school. Arithmetic was at one time taught in the college.

From another point of view several school people have charged that the elementary school is attempting to teach more arithmetic than anyone needs. Certainly, this charge is open to debate. No doubt certain schools are spending too much time on some phases of arithmetic, but the general achievement in a broad program of arithmetic is not satisfactory. This is attested by the cumbersome and frequently erroneous arithmetical procedures used by our public school graduates when they enter a simple vocation. We should not lose sight of the purpose for studying arithmetic, i.e., to educate boys and girls to become more self-reliant and to think and act more surely and correctly when they encounter a mathematical situation.

The amount of time devoted to arithmetic should depend upon the amount of arithmetic to be learned and also upon such psychological factors as the difficulty of learning. These are relative matters and as yet must be determined largely by personal opinion. In general, in conventional schools, the time devoted to arithmetic is approximately 10 per cent of the total school day (including play periods).

3. *If a child does not encounter arithmetic or see a need in learning it, then why should he bother to learn it? Should a child learn things in the elementary school when his greatest need or use of them will come in adulthood?*

All children encounter arithmetic even though they may not recognize it. Is it not true that we sense and appreciate better the things we have studied and understand? In 1928 when everyone seemed prosperous, a matron said to her child's nursemaid, "Just see that Bobby has the rudiments of walking; he will always have plenty of cars." In 1941 this same Bobby is walking to a public school. We should not be similarly shortsighted in respect to arithmetic.

If education could be continuous, it might be desirable to postpone many things now taught in the elementary school until the social or vocational need for them arose. However, it would seem somewhat unwise to postpone learning to swim until after the boat has capsized. A sensible procedure is followed in many schools in respect to percentage. In the upper elementary school grades the basic concepts, ideas, principles, and computations are learned in relation to situations that appeal to children of this age level; the more difficult and complex phases are omitted. With this background a pupil is prepared to meet the ordinary simpler uses of percentage and he has a reliable foundation upon which he may, at a later date, build more technical applications.

Society expects intelligent behavior from its members. A part of this behavior is dependent upon mathematical knowledge. When the whims of the individual conflict too strongly with the expectations of society, the individual should yield. This is a basic tenet in the compulsory attendance required in our public schools.

4. *How should the arithmetic content of the curriculum be selected? Should the content vary for different communities?*

A close relationship exists between textbooks and courses of study in arithmetic. Until about 1900 courses of study or curricula were very brief and frequently merely referred to pages of some particular textbook. Since 1900 a reversal has taken place and now textbooks are presumably based upon well-known courses of study. In fact, however, since the personnel of authorship of textbooks and curricula are frequently the same or at least represent the same point of view, a reciprocal relationship between curricula and textbooks exists. Since both textbooks and curricula are intended for use by teachers, they are usually prepared with a content and treatment with which many teachers will be sympathetic. Occasionally both texts and curricula are prepared in light of information collected from teachers in the field. Directly and indirectly the teacher exerts a considerable influence on both the content and the treatment of materials in a curriculum. He exerts a more individualized influence when he decides how he himself will teach and when he personally emphasizes certain phases of the prescribed content. In some schools the teacher is practically free to determine his own curriculum.

Several survey techniques have been employed by curriculum committees in an attempt to be more scientific in selecting the content of arithmetic. In general, the earlier surveys tried to discover what arithmetic was most commonly used by adults. More recent surveys have tried to classify the activities of children at various age levels. Two weaknesses are likely to appear in the survey of adult uses of arithmetic. First, both the surveyor and the one whose uses of arithmetic are being surveyed may not recognize all the mathematical situations and uses employed by the subject; and second, the surveyor and the surveyed may not sense all the potential functions of arithmetic that might be used if the person surveyed were better educated in arithmetic. This is apparent when one who knows arithmetic as a broad field observes an ordinary adult in both commonplace and novel situations. The survey of children's activities is subject to the same types of error.

A comprehensive survey of the actual uses and all the potential uses of arithmetic in the lives of people ranging in age from two years through adulthood would offer many suggestions to curriculum makers. Such a survey should include all phases of arithmetic in the cultural as well as the social and economic life of the individual. From the mass of data collected the curriculum worker would seek an organizing principle. The most apparent and probably the most reliable principle for educational purposes would likely be the logical relationships and sequential character of much of the data. Having determined what materials are desirable for basic objectives the curriculum worker would next study the functional relationships in order to determine a sensible teaching sequence. Textbook writers and curriculum makers frequently employ the results of informal studies of actual and potential uses of arithmetic.

Although the uses of mathematics may vary somewhat from one community to another, the basic mathematics involved is very similar. Hence, it is doubtful if arithmetic curricula should differ noticeably in content although they may differ considerably in method and in arrangement. Since the same textbook is often used in widely separated communities and since textbooks frequently become the actual curriculum, the same arithmetic content seems to suffice. From another point of view, producing a play on Broadway requires the same basic additions, multiplications, and percentages as producing a crop of wheat in the Dakotas. The mathematical concepts may be different in the two fields of production, but both fields require reasoning and judgment and many other types of mathematical thinking.

The selection of content for "activity-experience" schools is approached somewhat differently from the procedures used in more traditional schools. For the activity-experience school it is particularly important to formulate the aims and objectives in respect to arithmetic even if arithmetic is considered of secondary importance to the "activities" and "experiences." It is not sufficient to say that a child will learn the arithmetic which he encounters in a study of transportation or some other activity. As has already been pointed out, neither the child nor the teacher may be particularly alert to the arithmetic in the situation. For this type of school a chart of sequential learnings in arithmetic may be prepared so that teachers may know whether or not it is wise to pursue some particular phase of mathematics. Such a chart is useful also when it is found advisable to make an inventory of the pupils' arithmetical learnings. If it is found that the regular "activity-experience" program does not provide for all the desired learning of arithmetic, then a special program in arithmetic should be provided.

Arranging and placing the arithmetic content. The arranging and planning of the arithmetic content present these problems.

1. *Have logical and mathematical considerations played too prominent a role in the organization of arithmetic curricula?*

A curriculum that is organized in terms of the computational skills of arithmetic is necessarily a logical one. Most teachers realize that the process of division, for example, depends upon the processes of addition, subtraction, and multiplication, and therefore division is sensibly learned after the other processes. However, it is possible to begin division before all the usual abilities with the other processes have been achieved. The arrangement of a desirable sequence for teaching arithmetic must involve the logical dependencies of one process upon another. Other phases of arithmetic, as for example concepts, information, and principles, are usually related to processes and to that extent they should be studied in their mathematical relationships. Frequently greater meaning and significance are achieved when a phase of arithmetic is studied in relation to the social or economic situation in which it commonly occurs.

The topic of percentage will serve to illustrate how and when logical relationships are useful in developing a new process. To find a per cent of a number, most people convert the per cent rate to a decimal or to a fraction and then proceed to multiply. Hence it is necessary that work with fractions or decimals must have preceded this phase of percentage. Furthermore, since certain exercises dealing with this type of percentage are simpler when done with fractions and others are easier when done with decimals both methods should be learned. Other process phases of percentage illustrate this same mathematical dependence upon fractions and decimals. Computations with decimals follow the same pattern as the corresponding computations with whole numbers. In this way, by tracing the mathematical dependencies, a teaching sequence is developed. On the other hand, the topic of percentage also illustrates how some phases of arithmetic need not follow the same sequence as the process phases. The concept of percentage may be rather well developed with only a small knowledge of fractions and no knowledge of decimals. For example, the fourth grade boy who has learned that 4 per cent means "four out of a hundred" and who then reasons that this is the same as "two out of fifty" and "one out of twenty-five" and "eight out of two hundred" has grasped the fundamental idea of percentage.

Schools of the "activity-experience" type frequently get into difficulty when they ignore the logical relationships of arithmetic. Consider the predicament of the supervisor from such a school who said, "Of course I would teach long division in the second grade if the pupils encountered a need for it." Planning when to teach a topic is like bidding in contract bridge; one must know when not to bid as well as how and when to bid.

Logical relationships frequently facilitate learning and hence become elements of the methods of teaching. In general, whenever the logical relationships of mathematics aid, better than any other means, in the development of meanings and understandings and in the facilitation of learning, then the logical relationships should be employed. In actual school practice it is found that some teachers prefer and excel in one approach while other teachers prefer another. Most teachers use a combination of methods and devices to ensure that their pupils develop meanings and at the same time grasp the social and economic significance of their learning. Dr. Thiele states in his chapter dealing with arithmetic in the primary grades that the number system and number meaning are developed by using logical relationships together with concrete experiences with numbers.

2. *Are there age levels at which children learn certain types of arithmetic better than at other age levels?*

We have no evidence to show that a nine-year-old child will learn, for example, multiplication facts more readily than an eight-year-old child or a ten-year-old child, when age alone is considered. Other factors such as previous experiences with numbers and associations with the idea of multiplication are more important. Mental maturity, which is a function of both age and experience, bears some relationship to ease of learning. We have numerous instances to show that a twelve-year-old child can learn a given amount of arithmetic in a fraction of the time that is usually required for an eight- or nine-year-old child. Similarly ten-year-old children who have not previously attended school have learned to read as well in six months as six- and seven-year-old children learn in two years. The implications of these cases are interesting and need to be studied by curriculum makers. So many factors enter into learning that it is difficult to draw valid conclusions in relation to a single factor.

3. *Should arithmetic be taught in the kindergarten and in grades one and two?*

Whether or not the school plans to teach arithmetic in these grades, the children will learn it. Even at the ages of two and three years, normal children are developing ideas of size, amount, and number, and are making visual and mental as well as manual comparisons. Similarly in the kindergarten concepts of size, of shape, of amount, and of number are being developed in relation to the things which the children see and handle. These concepts and associations with them precede the stage when it becomes necessary to read and write figures. Opportunities for thinking and for the exercise of judgment frequently occur in the kindergarten. The wise teacher capitalizes these with the pupils. For example, instead of directing several pupils to get one large and two small mats for the playroom, the wise teacher will let the children experiment with the size and shape of rugs. In that way mathematical ideas of size and shape are fostered.

In many schools number records are started in the first grade and counting with meaning is developed. In many cases the arithmetic is based upon the normal experiences of the children around the school. The role of the teacher is to recognize the uses of numbers and assist his pupils in the formation of concepts and of principles of numbers and the number system. While pupils begin to organize and to systematize their learning at an early age level, mathematical sequence in learning is probably less important here than at later stages. A great deal of information, particularly about measures and their uses, is usually learned in grade one. Definite goals for attainment in this grade need not be set. However, it is desirable to record the arithmetical learning of the pupils at the end of the grade.

Practically all children who enter grade two have already learned a number of addition and subtraction combinations even though none were previously taught. Many teachers chart the addition combinations that have been learned and then proceed to develop new combinations through association with those already learned and with the aid of concrete materials. Information, concepts, and mathematical principles are involved in many of the affairs of children in the second grade and hence are available for study. A description of the arithmetic learning of children at this age level is given by Agnes Gunderson in *The Mathematics Teacher* for January, 1940.

A systematizing of learning appeals to children in the second grade. It appeals to their intelligence and helps them to sense the relationships and importance of what they are doing. A program that begins to organize the learning of arithmetic need not be formally carried out. In fact, the approaches to learning and the methods by which it is achieved may be quite as important as the usual goals. The writer believes that broad and definite goals in arithmetic should be attained in grade two.

4. *Is it possible to arrange a curriculum in terms of successive units or activities such as the post office, transportation, communication, and providing shelter? Will such a curriculum assure the learning of arithmetic?*

Such curricula are in operation in a number of schools and to some school people they are entirely satisfactory. It is a question of aims of education. Whether arithmetic can or should be learned in such a curriculum is another matter. Much depends upon the teacher. He may find it entirely possible and even very favorable to teach arithmetic by such a curriculum in the lower grades.

A topic such as communication may be fitted into a curriculum for five-year-old children or it may be suitable for fifteen-year-old children or twenty-five-year-old adults. Its place in a curriculum sequence depends upon the basic subject matter involved and what is done with it. From the mathematical point of view the topic might be limited to a study of the size, shape, weight, and postage of letters. On the other hand bracket functions, the principles of curve fitting, and differential and integral calculus may be needed if one wishes to study the more complex phases of communication.

The broad aims of arithmetic as previously described in this chapter cannot be achieved in the "activity-experience" treatment of topics such as transportation unless definite provision is made for systematic instruction in arithmetic. This is true particularly when processes are being learned. It is true also if the modes of learning are considered important in the aims or goals of learning.

5. *How should the grade placement of the subject matter of arithmetic be determined?*

This question is complicated by many factors which bear upon the curriculum, factors which frequently are solely matters of opinion. The nature of the curriculum, of the organization of the school, the scope and function of the curriculum, the methods of teaching, the modes of learning, and the promotional policies of the school, all affect the problem of grade placement. All the factors which influence the efficiency, rate, and amount of learning indirectly affect grade placement. For example, a school that has five twenty-minute periods per week devoted to arithmetic in grades three and four will not normally achieve as much as a school having the same number of thirty-minute periods. This affects grade placement in subsequent grades. Likewise, a school which promotes all its pupils annually on a basis of chronological age is likely to have an accumulation of difficulty in arithmetic and will need a placement in certain grades different from that of the school that holds its pupils to certain definite standards in each grade. Arithmetic learning accumulates and any force that affects this accumulation in turn affects grade placement. Grade placement is in fact a secondary problem, secondary to the one of discovering a good teaching sequence.

In most cases personal opinion has determined the grade placement of the materials of arithmetic. Personal opinion has value particularly if it is tempered with knowledge and experience and then further tested by the experiences of many other competent workers in the field. That was the procedure used in preparing the chart of grade placements which is found in the New York State Syllabus (1937). Because of the many factors that enter into grade placement, it is doubtful if at the present time opinion should or can be replaced by research. To illustrate, research may determine the lowest age or grade level at which an arithmetic topic can be learned readily, but we may not wish to place the topic at that level. Likewise, research may seek to determine the optimum mental age for learning a topic but that criterion alone is not sufficient. People do not agree on the interpretation of "optimum mental age" nor do they agree on how to use optimum mental age if it were found. This is illustrated by considering division. Simple division problems can be solved informally by pupils who have learned multiplication facts in the third grade. But certain exercises with two-figure divisors prove difficult for seventh and eighth grade pupils. The curriculum maker might wish to spread division over five school years in order to adjust the materials to the difficulty of the work, as he interprets this difficulty, but he soon realizes that other considerations are worth noting. The inherent relationships of division suggest that it is a "unit topic" and perhaps should be so considered in the curriculum. Also, division is a very useful process which if delayed until grade seven would hamper the activities of the learner. Obviously the curriculum maker must weigh factors and use his judgment.

An intriguing approach to the problem of grade placement is through a study of pupils' interests and activities to see whether certain interests persist for children of a given age. If positive results were obtained in such an inquiry, they would have to be carefully scrutinized by the curriculum maker in order to see if the mathematical content of the successive interests of the pupils was related to the necessary sequential phases of arithmetic. Arithmetic is the one school subject that is full of logical relationships and mathematical dependencies that facilitate learning and give meaning to it.

Researches dealing with phases of the problem of grade placement are not lacking. They have suggestive value for the curriculum maker but none of them have attacked the whole problem. Many of these research studies have been classified by Brueckner in his chapter, "The Development of Ability in Arithmetic," in the Thirty-Eighth Yearbook (1939) of the National Society for the Study of Education. Using these studies in part, Brueckner has set up "stages of growth" in arithmetic and the corresponding grade level at which these stages are reached. It is interesting to note that Brueckner's "stages of growth" are in reasonable agreement with current curriculum practice.

In the same Yearbook, Washburne discusses "The Work of the Committee of Seven on Grade Placement in Arithmetic." The Committee of Seven sought to determine optimum mental ages at which children can learn various arithmetical processes. As pointed out by Washburne the conclusions of the committee should be checked by others. In checking these conclusions with pupils in New York State the writer has found variations as high as one and one-half years from the grade placements suggested by the Committee of Seven. These discrepancies do not invalidate the work reported by Washburne because the two studies differed in scope, in methods of teaching, in tests for evaluation, and probably in many other respects. The important point to note is that any of these research conclusions that are based upon the learning of children are valid only for the specific factors involved in the particular research. Factors which define a study of this type and which limit its conclusions are: (1) the interpretation of descriptive terms such as "optimum mental age," (2) the amount of previous learning of the pupils, (3) the methods of learning previously used by the pupils, (4) the methods of teaching used in the research, (5) the avenues and materials of learning employed by the pupils, (6) the amount and distribution of time, (7) the measuring instruments and the way in which they are used, and (8) the interpretation of results. If a curriculum committee wishes to use the results of a study based upon the learning of pupils, it should also expect to use all the factors and interpretations employed in the study.

As stated at the outset, the problem of grade placement is very complex. It involves many factors that operate simultaneously. It would be a mistake to determine a grade placement upon any single issue. Curriculum committees should consult the available research but finally they will have to settle by opinion and experience the most important questions involved in grade placement.

6. *What factors are important in selecting a desirable arrangement for teaching arithmetic?*

The most important factor to be considered in selecting an arrangement or sequence for teaching arithmetic is the mathematical relationships which when used make it easy to progress from one phase to another. In the main these relationships are found in two categories: (1) those within a topical sequence, and (2) those that join one topical sequence to another. To illustrate the first with the topic of addition, in the early stages a child develops ideas of combining and he learns to combine small amounts by using real objects and by writing numbers and combining the numbers. Based upon these early ideas and abilities it is easy to progress to harder number combinations, to "decade addition," to simple column addition, and finally to the addition of several columns with "carrying." Even within the arrangement given here it is possible to make several slight shifts. But the various abilities or stages of learning addition are functionally related and these relations should be used in teaching because they give meaning to the process. The second type of relationship, that which joins one topical sequence with another, is well illustrated by division. The process of division actually uses addition, subtraction, and multiplication. These three processes do not occur in exactly the same way in which they occur in the ordinary exercises of addition, subtraction, and multiplication but, having learned them previously, a child readily uses them in the new operation of division. It is sensible to use these relationships when planning a sequence for teaching.

The non-process phases of arithmetic are also related but to a much less degree than the processes. Logically one might assume that *inch* should be learned before *foot* and *yard* but this is not necessarily so. Classification of measures according to use is a more natural learning procedure and teaching sequence than classification in relation to size. The comparison of measures, however, depends upon previously developed concepts of the measures involved and if the comparison is to be an exact one it depends also upon a process with numbers. Teachers who are acquainted with arithmetic readily distinguish the necessary dependencies which facilitate the learning of arithmetic.

Studies of relative difficulty of operations have been used to arrange a teaching sequence from the easy exercises to the more difficult. As McConnell points out in his chapter dealing with the psychology of learning arithmetic, these studies of "error" and of "difficulty" are founded upon assumptions and methods of learning that are questionable. In general, if no other factors are involved, a teaching sequence might follow the order of difficulty. Teachers will note, however, that the order of difficulty found in an experimental study represents a statistical average and that many of the pupils in any particular classroom deviate widely from the statistical average.

In arranging the teaching sequence, other factors such as "interest appeal" of materials and "stages of growth" of pupils should be considered, to the extent to which they have been investigated and found reliable. However, if such factors conflict with the necessary mathematical dependencies the latter should be selected as the organizing principle.

Instructional problems that affect the curriculum. Among these problems are the following:

1. *How do different instructional procedures affect the curriculum?*

Since different instructional procedures result in different kinds, amounts, and rates of learning, they must affect the kind of arithmetic, the amount of arithmetic, and to some extent the placement of the arithmetic in the curriculum. Because of the cumulative nature of arithmetic learnings, instructional procedures probably have a greater effect upon the arithmetic in intermediate grades and upper grades than on that in the lower grades.

The job, then, of the curriculum maker is to find the type of teaching procedure that best fits the materials of arithmetic and the age levels of the pupils. Probably rather different procedures should be used in grades two and seven. The whole matter of individual differences in both pupils and teachers enters into the choice of method.

2. *What values are attached to the practice of "stretching" topics over several school years?*

Our earlier arithmetics were arranged topically and logically and for many years a topic such as fractions or division constituted the major work for a school year. Two criticisms of this practice were raised: (1) lack of continued work with a topic resulted in forgetting, and (2) while some phases of a topic were easily learned others proved very difficult. In order that the difficulty of learning materials might better fit the abilities of the pupils, schools have gradually spread or stretched the content of topics so that at the present time study of the topic of division is spread over several years. This can be done without violating necessary logical sequences. Many teachers feel that this practice results in better learning and happier children. Obviously, however, the practice of "stretching" topics should not be carried to extremes. For example, it might be found in an investigation of difficulty that certain division exercises with two-figure divisors proved more difficult than similar exercises with three-figure divisors but that fact alone should not place them in different grades or the one after the other.

3. *What curricular arrangements can be made in order to provide for individual differences among pupils?*

Ideally, every pupil should advance at his own best rate of learning. Practically, however, that is difficult to arrange in a public school when thirty or more pupils may be assembled in a single classroom. Teachers are familiar with plans for individualizing instruction, as for example the Dalton Plan and the Winnetka Plan. The test of any plan or organization of this type is in the effect it has on the behavior and learning of the pupils.

In larger schools where pupils may be placed in "ability groups" the instructional job of the teacher is somewhat simplified. The pupils in the higher-ability group may progress farther, faster, and by different methods than the slower pupils. However, within any group of pupils, differences of kind as well as of amount and degree of intelligence will be apparent.

The policy of grouping pupils according to chronological age, together with the practice of promoting all who are socially adjusted within the group, has made it exceedingly difficult for teachers to maintain reasonable levels of achievement in the school subjects. This situation can be resolved only in terms of the aims of education. Teachers should not expect to achieve goals that the school officially neither expects nor provides for. All too often teachers are disheartened when a superintendent desires goals such as cooperation, adjustment, happiness, while the parents expect their children to learn reading, writing, arithmetic. This is a curriculum problem each community must solve.

Some schools are meeting the problem of individual differences through individualized instruction with the aid of printed materials such as workbooks. In the hands of a skillful teacher who has a relatively small number of pupils, this plan is successful. It must be noted that printed materials are not real objects and the situations provided in print are only vicarious. The typical workbook is particularly suitable for the drill and practice phases of learning arithmetic. While the child is in school he has a right to receive the best materials and avenues of instruction that are available. The teacher should assume his role as a guide and teacher and not delegate his greatest opportunities to a workbook. Pupils who progress without frequent guidance by a teacher frequently learn a wrong method instead of a correct one.

The whole matter of individualizing instruction might better be viewed as a problem of individualizing learning. A curriculum might provide for different amounts and different kinds of learning for various types of pupils. The final success of any plan of individualizing learning depends upon the teacher.

4. *The "activity-experience" curriculum has proved very valuable in learning the social studies and the language arts so why not adopt it for arithmetic?*

Many school people would deny that the "activity-experience" curriculum, as defined in this chapter, is entirely successful in the language arts and in the social studies. No doubt this type of curriculum has great value. The big problem today is to discover the types of learning for which it is best suited. Certainly it is very bad reasoning to conclude that if a plan of instruction works well for one field it is equally good for another.

The "activity-experience" curriculum appears to be better adapted to such phases of arithmetic as the development of certain concepts, the appreciation of the uses and significance of mathematics, and for opportunities for reflection, than it is for other phases such as the development and learning of a computational process. This type of curriculum seems also to be better adapted to the kind of arithmetic usually found in lower grades than it does to the typical arithmetic of the intermediate and upper grades. Again, we should seek an adjustment that combines the values of both traditional education and "activity-experience" education.

BUILDING A CURRICULUM IN ARITHMETIC

It is not the purpose here to give detailed steps for developing a curriculum or course of study in arithmetic. Rather, some of the major implications from the previous discussion will be summarized.

Type of school. First of all the community should decide upon the type of school organization and the basic pattern of learning that are desired. These two elements go hand in hand and should be outgrowths of the basic educational objectives. Because these matters are so important and affect so many people in different ways it is suggested that the teachers, parents, and school officials share in these basic decisions. Obviously, no school organization will attain a high level of achievement if the teachers are unsympathetic with the aims and procedures of the school.

Selecting curriculum workers. The personnel of a curriculum committee should be sympathetic with the general aims and with the proposed organization of the school. It should be competent by education and experience to work on an arithmetic curriculum. The two phases of competence that are most desired are (1) an understanding of children and how they learn, and (2) a broad knowledge of arithmetic as a field of learning which functions in social, economic, and cultural affairs. It is also desirable that the personnel possess a temperament which enables the members to work cooperatively on the various phases of curriculum construction. Previous experience in curriculum making may be valuable but is not necessary. Graduate study in "education" may be either a help or a hindrance. The value of an education derived from experience in the field should not be overlooked.

Form of the curriculum. Early in the process of curriculum building, it is desirable to visualize the form and the special features desired for the curriculum. Later developments may alter early decisions. Contemporary curricula are of different type and format. There is some evidence of a shift from the outline type of curriculum which features objectives and methods to a curriculum which is a "handbook" for teachers and which is full of suggestions and guidance.

A curriculum committee might study several recent curricula in arithmetic to note the organization and treatment of the content and also to anticipate the many different kinds of questions that arise as work progresses. The recent arithmetic curricula of New York State and of the City of Chicago may be consulted.

Types of questions that arise. As a curriculum committee proceeds, many questions arise. The committee may wish to use the results of research. But research conclusions do not agree. Hence someone will need to interpret and evaluate research. The committee may wish to call in "outside experts." Specialists in arithmetic do not agree on many specific matters and hence the committee must select those whose judgment and counsel are trusted.

Many questions dealing with teaching procedures and with methods of learning special topics and processes will arise. These are as varied as, "When and how should we use the experiences of the pupils?" and "Shall we use additive subtraction?" The matter of method in subtraction is particularly important. In order to save confusion among the children all the teachers in a school and perhaps in a community should use the same method. In relation to division such questions as, "Should the full written form be used with one-figure divisors?" and "Should division be 'stretched' over four school years?" arise.

A committee working on an arithmetic curriculum may wish to designate types or degrees of mastery to be expected from the pupils. For example, all pupils might be expected to learn to add and to know when to add to a high degree of perfection. On the other hand, only the more able pupils might be expected to learn how to find the rate of interest charged for a purchase on the installment plan. The curriculum maker faces many difficult but interesting problems.

Chapter IV

ARITHMETIC IN THE EARLY GRADES

FROM THE POINT OF VIEW OF INTERRELATIONSHIPS IN THE NUMBER SYSTEM

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IT IS the purpose of this chapter to focus attention upon the idea that a well-rounded program of arithmetic teaching should be concerned with the interrelationships in the number system. Another chapter in this Yearbook (the one written by Brueckner) deals in particular with the social phases of arithmetic teaching.

Since 1920 great stress has been placed upon the socialization of arithmetic. More recently there has been an observable trend, in both theory and practice, toward the teaching of a meaningful arithmetic which seeks to help children to appreciate and utilize the interrelationships in the number system. In this program the social values of arithmetic have not been disregarded; in fact, they have been selected with greater care than formerly in so far as interest and comprehension of children are concerned. The significant difference between the program of arithmetic which finds support in this Yearbook and that of a decade ago is in the extent to which children see meaning in the numbers which they use and operate. The keynote of the new arithmetic is that it should be *meaningful* rather than *mechanical.*

In this discussion dealing with the interrelationships in the number system two principles will be used as points of departure. First, children should become acquainted with and use numbers extensively, both as oral and written records, to describe what is done about quantitative situations. Second, successful and efficient extension of number usage from the crude methods of early childhood to those of the competent adult can be facilitated by increased insight into the interrelationships of the number system.

Insight in this connection also implies a method of learning based upon active participation and discovery rather than upon passive acceptance of the skills of arithmetic.

How these principles of social usage and insight into the interrelations of the number system might operate in a program of arithmetic teaching in the lower grades of the elementary school is the text of this chapter. The reader is forewarned not to conclude that the interrelationships which are described are the only ones which children might perceive. Neither is it the aim of the writer to submit an inflexible body of content but rather to demonstrate how certain content may be taught with due consideration given to what Buckingham[1] termed "social significance," "mathematical meaning," and "individual insight." Arithmetic which is socially significant and mathematically meaningful for the individual student cannot be formalized.

THE CONTENT OF THE ARITHMETIC CURRICULUM

"Content," as here used, has a double meaning. It not only connotes skills and abilities which are selected as goals for arithmetic teaching but refers also to the interrelationships which serve to give the skills and abilities meaning. In the discussion which follows, consideration will be given to representative topics, the aim being to present a point of view with respect to arithmetic teaching.

Early grade number experiences. It has become customary to designate the early number experiences of very young school children under the heading of "number readiness." In the strict sense of the word the number learning of five- and six-year-old children cannot be separated from that which is acquired when the ages of seven, eight, and nine are reached. Likewise, the number knowledge of five- and six-year-olds is a refinement of still earlier ideas regarding quantity. Little consideration, however, has been given in our schools to the possibility of developing young children's number ideas in a systematic way. In most schools systematic instruction begins at the age of seven or eight.

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[1] B. R. Buckingham, "Significance, Meaning, Insight—These Three," *The Mathematics Teacher*, 31 : 330-335, January, 1938.

The so-called "readiness" instruction which precedes it is usually unplanned and undirected. Brownell's[2] study of the development of children's number ideas provides conclusive evidence to support the contention that the number learning of even kindergarten and first grade children should not be left to chance.

A kindergarten teacher was observed who endeavored to direct the development of number ideas of her children in a systematic way. In her room the leader of each table of six children had been in the habit of going to the cupboard within the room for boxes of crayon. In so doing he would look in turn at each child at the table as he selected the box of crayons for that person. The teacher, wanting to lead her children to higher levels of response, placed the crayons out of sight of the children in a small room adjacent to the kindergarten. Although the leaders experienced difficulty at first, they soon learned to count the children at their tables, including themselves, keep the numbers in mind, go to the next room, count out the required number of boxes, and return with them. The teacher varied the experiences of the children by asking them to obtain a given number of sheets of paper, count boys and girls present, set out a certain number of chairs, etc. Thus the development of number ideas was consciously directed by the teacher. Children were carried from the simple level of matching objects with children to that of dealing with total groups.

Even at this early stage in the school life of the child the two principles of arithmetic teaching proposed earlier were operative. Numbers were used as records to describe concrete objects and certain relationships between numbers were perceived. Obviously children on the kindergarten level used number names rather than number symbols to describe quantity. The relationships between numbers were those which enabled children to distinguish between groups, i.e., the cardinal number idea. In the operation of the average classroom there are countless opportunities for the development of the number ideas for which young children are ready.

The problem of induction into the use of number symbols does, however, seem to be difficult for many teachers and children alike. From the study by Brownell, to which reference has already been made, there is strong support for the contention that much experience with concrete numbers of the type described above should precede any attempt to deal with abstract number symbols. Brownell's conclusion was stated in these words:

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[2] W. A. Brownell, *The Development of Children's Number Ideas.* University of Chicago Press, Chicago. 1928.

"The theory presented and defended in this section is that success in making the transition from concrete number to abstract number is largely conditioned by the stage of development which has been attained in ability to deal with concrete numbers; that is, other things being equal, pupils who thoroughly understand concrete numbers are, because of that fact, unlikely to encounter serious difficulty in learning the additive combinations; conversely, other things being equal, pupils who have not developed very far in the ability to deal with concrete numbers are on that account almost certain to encounter serious difficulty in learning the additive combinations."[3]

Teaching children to use number symbols as means of expressing ideas is comparable to teaching them to write words. If children put together the symbols of the alphabet to name the things with which they have direct experience, it would seem that number symbols should be used in the same way. Thus the symbol 8 may be recorded to indicate the number of children in a row, the number of books on a table, the number of windows in a room, etc. Even a number combination such as $5-2=3$ may serve to describe what happened when 2 of 5 pennies were spent, 2 of 5 cookies were eaten, 2 of 5 marbles were lost, etc. The early study of number symbols is therefore more than an exercise in handwriting and number readiness; more than the repetition of number names in serial order; it is a study of ideas and ways of recording them.

The transition from visual oral to abstract written level of response. In a recent publication Morton[4] suggests that children pass through four stages of response before they reach the point of being able to deal with abstract number. The stages suggested by Morton and others are:

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[3] *Ibid.*, p. 125. [4] R. L. Morton, *Teaching Arithmetic in the Elementary School, I, Primary Grades*, p. 12. Silver Burdett Company, New York. 1937.

1. The object stage: purely concrete number.
2. The picture stage: pictures of familiar objects.
3. The semi-concrete number stage: dots, lines, circles, etc.
4. The abstract number stage: number symbols.

Advantage may be taken of the natural tendency of children to group any given number of objects in different ways to help them make the transition which is described above. Anyone who has watched children playing with blocks, coins, sticks, and other common articles, has observed children arrange and rearrange objects in many ways and groups. A teacher conducted such an activity in the following manner:

Four chairs were placed in a close row before the children. The pupils were asked what number should be written to tell how many chairs were in the row. The children agreed upon the number 4, and a 4 was written upon the blackboard by the teacher and by each child upon his paper.

The teacher then asked the children how they could picture the row of chairs with line drawings made like the printed letter h. From among several drawings the class selected one in which the 4 chairs were placed in a line with little space between the chairs. (h h h h)

Next the teacher suggested that the 4 chairs could be arranged in other ways and still be in a straight row. A child volunteered to place the chairs in another way and separated them into two groups with a wide space between each group. Following the suggestion of the teacher a drawing of the new arrangement was made upon the blackboard and upon the children's papers. It obviously was h h h h.

Other ways of placing the chairs in two groups were worked out with the chairs and drawings made to indicate arrangements like h h h h and h h h h. The activity resulted in a final record such as:

4
h h h h
h h h h
h h h h
h h h h

To check the understanding of what had been done, children were asked to move the chairs to show what different parts of the record meant.

This was followed with making arrangements and picture records of other numbers of chairs and of children and of books. Soon the teacher raised the problem of making complete records without any manipulations.

When the children understood what was meant by arranging objects into two groups, and could picture the arrangements, the teacher suggested that crosses such as X could be used instead of the drawings of chairs, children, and books. Thus the representation of 5 was changed to:

5
XXXXX
X XXXX
XX XXX
XXX XX
XXXX X

Again children were asked to explain different parts of the pictures. They soon reached the point of being able to describe parts of the "5 record" with such remarks as: "All five may be close together," or "You may have one out on one side and four on the other side," etc.

Finally the children were led to substitute numerals for crosses. A number such as 6 was then analyzed and the analysis recorded in the following manner:

6
XXXXXX
1 and 5
2 and 4
3 and 3
4 and 2
5 and 1

The activity which has been described terminated when the numbers up to 9 had been analyzed on the abstract level.

In this activity new learning started with real experiences. These were recorded. The activity brought to light many number relationships. Finally the activity ended with a consideration of relationships on an abstract level. From another point of view a transition was effected from a verbal to a symbolic method of describing situations which involve quantity.

Combining and separating numbers. In the normal progress of the child through school a point is reached at which his efficiency in arithmetic is impeded if certain number relationships are not understood and known to the point of immediate recall. Reference is here made to what are commonly termed the addition, subtraction, multiplication, and division combinations. The National Council Committee on Arithmetic[5] took the position in its preliminary report that the school should teach the facts and skills of arithmetic in a systematic way. It recognized the place and value of incidental learning but did not favor programs of teaching which stopped short of fixing for retention what has been learned. The Committee, on the other hand, looked with disfavor upon a systematic method of teaching which did not give due consideration to meaning and understanding of what is learned.

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[5] R. L. Morton, "The National Council Committee on Arithmetic," *The Mathematics Teacher*, 31 : 267-272, October, 1938.

Systematic study of the number combinations. The methods advocated for the mastery of the number combinations in a systematic way fall into two general classifications. One of the systematic methods of teaching the number combinations is based upon the theory that after children have become acquainted with them, the single combinations must be repeated over and over again until they are "learned." Brownell[6] classified this type of learning under the heading of "Drill Theory." The objectionable features of the "Drill Theory" when exclusively relied upon are too well known to require enumeration.

The second method which has been more recently advocated and used in certain places is in keeping with what Brownell termed the "Meaning Theory." In practice this method features the "cultivation of comprehensive general ideas" in the sense suggested by Judd.[7] Those who are in accord with the "Meaning Theory" have found different ways of applying it to the teaching of the number combinations. The differences are mainly in types of "comprehensive general ideas" or generalizations children are led to utilize. A discussion of representative plans follows.

Generalization in teaching number combinations. The generalization which has been most widely employed in teaching the number combinations is that numbers may be analyzed and synthesized with the aid of tens. Wheat,[8] Badanes,[9] Morton,[10] and others have advocated the teaching of additions whose sums are larger than ten by teaching children to rearrange two numbers into a ten and so many more; for example, $8+5$ may be found by thinking $8+2=10$ and 3 more makes 13. Thus thirty-six combinations may all be linked up with one generalization, namely, any two numbers the sum of which is greater than 10 may be regrouped as a 10 and so many more. Wheat would apply the method of tens to the learning of the subtraction, multiplication, and division combinations to which the ten's generalization will apply. For example, the child may obtain the product of 6×4 by grouping four 6's to make two 10's and 4. This represents a point of view which utilizes a single generalization very extensively.

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[6] W. A. Brownell, "Psychological Considerations in the Learning and Teaching of Arithmetic." *The Teaching of Arithmetic*, Chap. I. Tenth Yearbook of the National Council of Teachers of Mathematics, 1935.

[7] Charles H. Judd, *Education as Cultivation of the Higher Mental Processes*. The Macmillan Company, New York. 1936.

[8] H. G. Wheat, *Psychology and Teaching of Arithmetic*. D. C. Heath and Company, Boston. 1937.

[9] Julie E. Badanes and Saul Badanes, *A Child's Number Primer*. The Macmillan Company, New York. 1929.

[10] R. L. Morton, *Teaching Arithmetic in the Elementary School, I, Primary Grades*. Silver Burdett Company, New York. 1937.

In the teaching of the addition combinations other writers advocate a plan of grouping and regrouping the single numbers from 2 through 18 for the purpose of building generalizations about each number. For example, 11 would, according to this plan, be regrouped as $10+1$, $9+2$, $8+3$, $7+4$, $6+5$, and their reverses. Although in contrast with the idea of employing the ten's principle, the plan of analyzing and synthesizing each number seems to represent a somewhat restricted use of "comprehensive general ideas." Analyzing and synthesizing are valuable in that pupils learn about groupings of numbers with which they will deal later.

Another method to which approval has been given is that of teaching related addition and subtraction and corresponding multiplication and division combinations at the same time. Thus $5+3$, $3+5$, $8-3$ and $8-5$, as well as 4×2, 2×4, $8\div2$ and $8\div4$, would be centered around the study of the number 8. The relationships thus established are of great value.

Thus far the generalizations which have been given consideration include those which

(1) Utilize tens for purposes of analyzing and synthesizing numbers.

(2) Indicate relationships among the combinations into which single numbers may be analyzed.

(3) Are based upon an awareness of interrelationships between the processes of arithmetic.

There is little experimental evidence which indicates the superiority of use of one type of generalization over another. In a study by Thiele[11] an effort was made to teach the addition combinations alone according to a plan which aimed to give children experiences with many addition generalizations. For example, in this study children were directed in their discovery of generalizations with regard to the following:

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[11] C. L. Thiele, *Contribution of Generalization to the Learning of the Addition Facts.* Contributions to Education, No. 763, Bureau of Publications, Teachers College, Columbia University, New York. 1938.

(1) Adding 1, 2, and 0 to any number within the comprehension of the children.

(2) Combinations related to the same number.

(3) The reversals of combinations.

(4) The relationships existing among the doubles of numbers.

(5) The relationship of certain combinations to the doubles, i.e., 8 + 7 to 8 + 8.

(6) The relationship of 10 to the teen numbers.

(7) Making tens and so many more.

An organization of the addition combinations was used in this study which directed the pupils in the discovery of useful generalizations. Pupils in turn dealt with situations in which there was a need for adding 1, adding 2, adding 0, like numbers, numbers almost alike (3 + 4, 4 + 3, 4 + 5, 5 + 4, etc.), adding to 10, adding to 9, adding pairs of numbers the sums of which are greater than 10 or less than 10.

The generalizations were not formulated for the pupils but in their own words they indicated such discoveries as—

> "When you add 1 to any number the answer is the next higher number."
> "You go up two numbers or skip a number when you add 2."
> "The number is the same when you add 0."
> "You can turn any combination around."

As the consideration of the addition combinations proceeded, no attempt was made to cause the children to use specific generali- ations. The aim of the teachers was to promote the thinking of swers, i.e., seeing relationships rather than counting or guessing. ie combination 8 + 6 was met late in the course. Children left their own resources suggested several ways of obtaining ie sum. For example, they said, "8 + 6 = 14 because 6 + 6 = 12 and 12 + 2 = 14," "8 + 8 = 16 and 16 − 2 = 14," "8 + 2 = 10 and 10 + 4 = 14," and "8 + 6 is the same as 7 + 7."

Attention should be called to the fact that the teaching procedure was planned in such a way that the pupils first made records of activities which involved social experiences. They manipulated concrete materials as a means of obtaining sums until they were able, under the direction of the teachers, to discover useful generalizations by means of which the sums could be obtained directly.

It should also be pointed out that the pupils were required to carry their practice with the combinations to the point that they could give answers readily. Timed tests were employed for purposes of increasing the speed of response to the combinations. The remedial work differed from that usually employed in classrooms in that restudy was never on a single combination but for the purpose of recalling a useful generalization and reviewing it.

The results of the study described were in keeping with a previous study by McConnell.[12] The children who dealt with interrelationships within the number system in their study of the addition combinations surpassed children who studied the combinations by a method of repetitive drill in which no attention was paid to interrelationships.

The issue at present is not whether the use of certain interrelationships produces better results than the use of other relationships. The issue is whether schools should continue to neglect relationships or utilize them in their teaching of arithmetic. The evidence seems to be in favor of the latter.

Interrelationships among subtraction, multiplication, and division combinations. Space does not permit a detailed discussion of the teaching of the subtraction, multiplication, and division combinations. At an earlier point subtraction generalizations were described. They included the ideas that certain addition combinations are related to certain subtraction combinations, and that certain numbers can be analyzed into tens and so many more. Illustrations were given of interrelationships between multiplication and division combinations and between all types of combinations into which certain numbers could be analyzed. Among the combinations of each process there are other interrelationships which may be brought into a program of teaching.

Subtraction interrelationships. The records of subtraction situations may also be utilized for the purpose of stimulating children to discover interrelationships between the minuend and difference when the numbers subtracted are either 1, 2, 0, a number equal to, one less than, or one-half of the size of the minuend. Reference has already been made to interrelationships between minuends larger than 10 and differences less than 10 which involve the ten's idea. In the use of these interrelationships the difference between two numbers such as 15 and 8 could be found either by first subtracting 5 and then the remaining 3 from 10, or by thinking 8 and what makes 15 in the form of $8+2=10$, $10+5=15$, therefore $15-8=7$.

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[12] T. R. McConnell, *Discovery vs. Authoritative Identification in the Learning of Children.* University of Iowa Studies in Education, Vol. IX, No. 5, September 15, 1934.

The reader is again admonished not to infer that children who "think" answers in the initial stages of learning subtraction combinations, guided by the perception of interrelationships, must of necessity continue to do so. As in the case of learning the addition combinations, application and practice may be introduced and so directed that automatic response is achieved. The process is a long one requiring judgment on the part of the teacher regarding the progress of children, as well as purpose on the part of children to master subtraction combinations. It is conceivable that some time in the future children will be allowed two or three years rather than a semester or two for the mastery of number combinations. The process of learning, according to this conception, is one of progression from relatively crude but meaningful procedures to a final stage at which a pupil is able to give the answers both readily and accurately.

Generalizations among multiplication facts. Although the multiplication combinations have been taught for many years in the form of "tables" or families, little recognition has been given to the possibility of utilizing the relationships which are inherent in each so-called "table" among the "tables." It has been common practice to introduce sets of multiplication facts by demonstrating the repetition of equal units. However, this type of instruction has been and is followed in most schoolrooms by repetitive drill on the tables. In general little time or effort is used to cause children to build up their own "tables" and from examination of each "table" to note useful generalizations about them.

In the teaching of the multiplication combinations which are necessary for ordinary purposes, two objectives stand out. First, the concept of multiplication as a quick way of grouping equal units must be sensed. Second, the separate combinations must in time be habituated for the sake of efficiency and economy for later computation with numbers. It is in the perception of the multiplication concept that social usage plays a part. However, as is pointed out in the chapter dealing with materials and devices for teaching, concreteness of experience is an important characteristic of learning situations in which new concepts are formed. Thus the multiplication concept seems to grow out of situations in which equal groups of concrete objects are placed together. The school aids children in making symbolic records of these regroupings so that they in turn may make symbolic records when situations are described or imagined. The extent to which the regrouping of objects is carried on for real or simulated purposes is determined by the point of view held toward teaching in general. The strengths and weaknesses of various programs of education are briefly discussed in the chapter on curriculum problems by Sueltz. Regardless of the point of reference, it is commonly agreed that children should know certain multiplication combinations.

The proposal is made that by focusing the attention of children in their earliest multiplication experiences on the social situation in which units of ten are involved, children may be led to perceive certain principles inherent in all "tables" or sets of related multiplication combinations. For example, there is a sequence of products from decade to decade which most children quickly sense in the table of 10's. The sequence may be followed in a descending, as well as in an ascending, order. It may begin with any combination and move in either direction, i.e., from 50 we may go to 40 which is 10 less, or to 60 which is 10 more than 50. Each combination has a reverse. Each product has a certain ending; in the table of 10's it is 0. The relationships between products is the same as between multipliers, i.e., the product of 8 × 10 is twice as large as that of 4 × 10. Each of these interrelationships or characteristics may, by direction on the part of the teacher, be brought out in the study of the table of 10's.

If the teaching of the table of 10's is approached from the point of view of having children think rather than memorize by a rote method, some time will be spent in examining the table. Under the guidance of the teacher, deductions such as those listed above may be made by children. The newly discovered interrelationships may in turn be used in other situations in which a knowledge of the table of 10's is required. They will serve to help children think products if they cannot recall them readily. These are not rules which children learn and apply in specific situations but rather interrelationships or generalizations which individual children select as a result of their experiences.

Although a high order of difficulty is ascribed to the table of 9's when taught by rote drill methods, the combinations of this table become comparatively easy to learn when taught meaningfully. After children have built up the table of 9's, they may be led to discover the descending order of the one's digits and the ascending order of the ten's digits in 9, 18, 27, 36, 45, 54, 63, 72, 81, 90. In building up the table by successively adding 9, they utilize the idea that the sum is one less than it would be if 10 were added. Children may be led to observe that the sum of the digits of each product is 9; thus 9 × 6 cannot be 56. Also the products of 9 and even numbers are even numbers. The first two or three combinations are readily recalled as reverses of easy combinations. Forty-five and 90 serve as reference points when near-by combinations must be "thought out." In fact, the table of 9's contains so many interrelationships that it may easily be vested with meaning.

The inference should not be made that it should be the purpose of the teacher to teach a set of generalizations in a formal manner. The purpose should rather be to guide children in the discovery of numerous interrelationships from among which individual children may select those which will best help them think answers in the initial stages of learning multiplication tables.

The kind of ability which should result from the type of instruction described above is well illustrated in the account of the boy to whom Gladys Risden refers in *Progressive Education,* February, 1940. The article is entitled, "What Price Mechanization?" The boy could not memorize the tables of 8 and 9. In an out-of-school situation he found the products with ease.

When asked how he found that eight 8's are 64, he replied:

> "I thought that out two ways. Two eights are sixteen, four eights would be two sixteens, and that would be twenty and twelve, and eight eights would be two thirty-twos, and that would be sixty-four, and a quicker way would be just eight less than seventy-two, and ten less would be sixty-two, so eight less would be sixty-four."

There are those who object to teaching which permits such thinking because they fear these roundabout methods will interfere with memorization. There is no evidence to support this contention.

Interrelationships in the processes of arithmetic. As children grow older, situations arise for which more than a knowledge of the simple number combinations of addition, subtraction, multiplication, and division is required. The characteristics of our number system which may well serve to unlock the mysteries of the processes of arithmetic for children are the following: the significance of ten as a base or as a standard group; the function of zero as a place holder; the principle of position or place value and the principle that groups—tens, hundreds, thousands, etc.—may be treated just as ones or units are treated. The importance of each of these concepts and how they may serve to give continuity to the study of the reading and writing of numbers, and to the study of the facts and the processes, will be discussed.

Number concepts and work in reading and writing numbers may emphasize basic characteristics of the number system. From the earliest work with numbers, the pupils' attention is focused upon the idea of groups. Pupils first become familiar with groups of things ranging from a single thing to a group containing many things. Pupils discover that a group of a given size may be rearranged in several different ways. For example, 8 may be grouped as 8 ones, or it may be separated into groups of 7 + 1, 6 + 2, 5 + 3, 4 + 4, 3 + 3 + 2, or as four 2's, etc. This early work really includes the development of certain easy addition combinations. From it many of the addition combinations may be learned.

When the numbers from 10 through 19 are first met for purposes of systematic teaching, regardless of the program of instruction in vogue, the special significance of the ten group both in writing and thinking these numbers may well be emphasized. When groups containing from 13 through 19 objects are counted, pupils may be directed to listen for familiar names in the numbers they are saying. To exaggerate this similarity the numbers may sometimes be called three-teen, four-teen, five-teen, six-teen, etc. Pupils soon sense the fact that the name of the ten group, changed slightly to say "teen," is heard in each of the numbers from 13 through 19, and that the other part of each word contains one of the familiar words three, four, five, six, etc.

Groups of objects such as tickets, jackstraws, books, or whatever objects are a part of situations under consideration, may next be consciously built up by starting in each case with a group of ten and adding the number of ones which is heard in the number name—18 is a 10 group and 8 ones. In everyday life such grouping is commonplace and therefore should find a place in the classroom even if it were not helpful in arithmetic training. Even though "eleven" and "twelve" do not seem to contain the words "one" and "two," it may be shown concretely that 11 equals a 10 group and 1, and that 12 equals a 10 group and 2. Thus advantage may be taken of natural experiences to lead children to make useful deductions about our number system which many of them would not ordinarily perceive. That this is the function of the school is taken for granted.

It is particularly important that those children who have learned oral rote counting before entering school be given many opportunities to develop an appreciation and understanding of numbers through a study of number uses and number relationships.

Equally as great importance attaches to 10 when pupils are first taught to write the numbers from 10 through 19. To write 10, pupils may note that we use two separate symbols—one of which looks exactly like the symbol used to stand for a single object. Likewise to write the numbers from 11 through 19 we use the same symbols used before but in a certain definite arrangement. Since pupils have built the numbers concretely to 10 by combining a group of ten with a certain number of ones, they may be guided to see that in writing these numbers the figure 1 in the ten's place means 1 ten group.

When the numbers 20 through 29 representing groups of familiar objects are analyzed and written, it becomes evident that here we use 2 ten's groups in combination with one or more units. As work in building numbers to 99 is continued, the teacher who is alert to the possibilities may continue to emphasize the importance of the ever-present ten's group. Children who have had no instruction have been observed in the act of grouping playthings in trains and piles of 10 and so many more.

A device sometimes used when objects grouped in tens are counted is to have pupils say two-ty, three-ty, four-ty, five-ty, etc., to make very evident that these numbers really mean two-tens, three-tens, four-tens, etc. It will be observed that in the consideration of numbers from 1 to 19, the order is objects—language—symbols, while from 20 to 99 the order is symbol—language—objects. Thus the ideas all cooperate; only the emphasis shifts.

While children deal concretely with the numbers to 100, a systematic number chart may be built like the one shown below. The teacher places it on the board and each pupil places a similar chart in the back of his notebook. Again the school is helping to organize learning which might otherwise remain unorganized. The point at which children are ready for this is a matter of grade placement.

1	11	21	31	41	51
2	12	22	32	42	52
3	13	23	33	43	53
4	14	24	34	44	54
5	15	25	35	45	55
6	16	26	36	46	56
7	17	27	37	47	57
8	18	28	38	48	58
9	19	29	39	49	59
10	20	30	40	50	60 . . . 100

As pupils are learning to use numbers to 100 in their thinking and writing, and after a number chart has been assembled, they may be guided to make some generalizations regarding the importance of the place in which a number symbol is written and regarding the function of zero as a place holder. The path of learning is and should be from experiences with concrete objects to number symbols by which experiences may be represented.

To focus attention upon the concept of place value, the pupils may be asked to show on the number chart all the numbers for which it was necessary to write the symbol 4. Pupils will note that the symbol 4 occurs in 4, 14, 24, 34, etc., through 94, and in all of the numbers from 40 through 49. The two different values indicated by 4 may be illustrated through the use of concrete materials. For example, if a situation involving tickets is the point of departure, 4 single tickets may be used to show the meaning of 4 when it is written alone or when it is written to the right of one other symbol. Four bundles of tickets, containing 10 tickets each, may be used to show the meaning of 4 when it is written to the left of one other number or at the left of a zero. Further study will reveal to pupils that the same thing is true of 7, or 5, or any of the other 9 symbols. This early study of place value will lead pupils to realize the difference in value between numbers such as 17 and 71, between 38 and 83, etc. Greater discrimination in writing numbers in columns for addition will also be developed. Exercises in writing and comparing numbers will help pupils to develop the ability to use the terms "one's column" and "ten's column" or "one's place" and "ten's place" with understanding.

To focus upon the use of zero as a place holder, attention may be directed to the numbers 90, 80, 70, and to the fact that they mean 9 tens, 8 tens, 7 tens, and not 9, 8, or 7 ones. Reference at this point may be made to the concrete objects grouped into tens. It may also be pointed out that when we write 91, 82, or 73, we have numbers which are made up of both tens and ones, so we write a number in both places. However, if we meant 9 tens, but wrote simply a 9, it would look as though we meant 9 ones, so we need something to write with 9 to show that it means 9 tens. This "something" we write is a zero. Thus zero is really used to indicate an absence of ones when a number is made up only of tens.

Dealing with numbers to 100 by the method suggested above may develop a consciousness of the special significance of the ten group. The necessity for thinking carefully about the relative positions in which the nine numerals are written when they are used to express numbers greater than 9 may also be appreciated as well as a tendency to regard zero as a symbol which helps the numbers from 1 through 9 to indicate that groups and not ones are meant. In short, such activities as these should disclose the rhythm of our number system.

In the next stage of developing number meanings from 100 to 1,000 these characteristics repeat themselves. They lead to a still fuller insight into the number system. When pupils assemble ten groups of ten each, they find that they are not named ten-ty, as might be expected, but are called by a distinct new name of "one hundred." When 10 tens are written, a third place becomes necessary, which is called the *hundred* place. Thus, 10 tens are grouped together to form a new standard group to help us in understanding all numbers larger than 99. Just as the group consisting of 10 ones was important as a measure of all numbers from 1 through 99, we now find that a group consisting of 10 tens is important as a measure for all numbers from 100 through 999.

As pupils develop a number chart by 1's from 100 through 200, the need for a zero to keep figures in their correct positions may again be emphasized. We write 101 which means that we had enough objects to make 1 group of 100 and 1 object over. However, to separate the 1 meaning hundreds from the 1 meaning "ones," we need a symbol, since we cannot write any of the figures from 1 through 9 in ten's place. A zero is therefore written to keep the ones of 101 in their proper places.

If the work of studying number meanings to 1,000 has been thoroughly done, many of the pupils will have sensed the rhythm and order of the system to the point that they will be able to write the numbers beyond 1,000 with almost no assistance from the teacher. However, the teacher may need to point out the fact that 10 hundreds form a new standard group and is given the name thousand, and that in the number 1,000 we can always see 10 hundreds or 100 tens. Obviously the extension of number meaning to 1,000 can only occur when children have dealt with numbers sufficiently to form concepts of larger numbers.

Understanding characteristics of number system gives meaning to the processes. Just as the idea of grouping in tens or powers of ten is fundamental to the development of the ability to think and record numbers, we find that it is also a basic idea in computation by the different processes, which may all be looked upon as fundamentally rearranging or regrouping procedures.

When the occasion arises requiring the addition of two-place numbers, pupils may be led to perceive that tens may be combined just as ones are combined. However, if combining ones provides enough or more than enough ones for another ten group, then this new ten group is transferred or carried to the ten's column and combined with the other ten's groups. It is this explanation which makes carrying appear as a logical, understandable procedure in contrast to the purely mechanical device which some pupils learn of always writing the right-hand digit of a two-place number in the answer and carrying the left-hand digit. Furthermore, if "carrying situations" are directly related to activities in which objects are actually manipulated, the process of carrying is meaningful rather than mechanical.

A few illustrations may be taken from each process, social settings for the illustrations being omitted.

(1) $\begin{array}{r} {\scriptstyle 1} \\ 48 \\ +39 \\ \hline 87 \end{array}$ Combining ones gives 17. This is enough to form another ten, and there will be 7 ones left. Write the 7 in the one's column in the answer. Combine the new ten group with the others of its kind and we have 8 tens. Write the 8 in the ten's column in the answer. At least during the early learning period it is advisable to indicate the carrying number. It is a definite part of a number record representing real experiences. Some children are able to complete the work without it.

(2) $\begin{array}{r} 728 \\ +174 \\ \hline 902 \end{array}$ $8 + 4 = 12$ ones. Change to 1 ten 2 ones. Transfer the 1 ten to the ten's column. Combine tens: $1 + 2 + 7 = 10$ tens. This is enough to make one larger group of 1 hundred. Combine the hundreds: $1 + 7 + 1 = 9$. When the given groups are combined and regrouped we have 9 hundreds and 2 ones. The vacant ten's column must be filled, so a zero is written in ten's place to keep the ones and hundreds in their proper places.

To deal meaningfully with the subtraction process pupils need to keep in mind the ten's relationship and to note that again tens and powers of ten may be treated just as ones are treated.

In addition we combine like groups and seek every opportunity to regroup into a larger group—ones into tens—tens into hundreds, etc. In subtraction it is often necessary to break up large groups into smaller ones. This is precisely what frequently happens when objective material is manipulated.

(3) $$\begin{array}{r} 15 \\ -8 \\ \hline 7 \end{array}$$

To subtract 8 means that we wish to find out how large the other group will be if we separate 1 ten 5 ones into 2 groups—one of which contains 8 ones. We may think 8 from 10 leaves 2. These 2 ones with the other 5 ones make 7 ones, so 15 − 8 leaves 7. It is also possible to think of the 1 ten and 5 ones as 15 ones and subtract directly.

(4) $$\begin{array}{r} \overset{4}{\cancel{5}}\overset{10}{\cancel{0}} \\ -38 \\ \hline 12 \end{array}$$

From 5 groups of 10 or 50 take 3 tens 8 ones. Regroup the 5 tens as 4 tens and 10 ones. This may be done concretely. From these 10 ones take 8 ones, leaving 2. Only 4 ten groups are left. Take 3 away and 1 will be left.

Several illustrations will serve to indicate how the basic principles enumerated for addition and subtraction apply to multiplication and division.

(5) $$\begin{array}{rl} 48 = & \text{4 tens} \quad \text{8 ones} \\ \times 6 & \qquad \times 6 \\ \hline 288 & \text{24 tens} \quad \text{48 ones} \end{array}$$

6 times 8 ones are 48 ones. Six groups, each containing 4 tens, make 24 tens. 48 ones may be regrouped as 4 tens 8 ones. Then we have 24 tens + 4 tens, or 28 tens. 28 tens may be further regrouped and written as 2 hundreds 8 tens and 8 ones, or as 288.

(6) $$\begin{array}{r} 26 \\ 6\overline{)156} \\ 12 \\ \hline 36 \\ 36 \\ \hline \end{array}$$

Here we have 1 group of 100, 5 groups of 10, and 6 ones. The numbers may represent tickets in bundles of the sizes designated. We wish to put the tickets into 6 equal piles. We cannot put a 1 hundred bundle in each of the 6 piles until we break the 1 hundred bundle up into 10 ten groups. Then we have 15 ten groups. This is enough to put 2 tens in each pile and we use up 12 tens leaving 3 tens. We may emphasize here that nothing need be written above the hundred's place, but 2 may be written over the ten's place of the dividend since we had enough tens to divide into 6 piles. Three tens is the same as 30 ones. 30 + 6 = 36 ones. From 36 ones we can put 6 ones in each pile, thus using up 36 ones.

(7) $$\begin{array}{r} 204 \\ 4\overline{)816} \end{array}$$

To regroup 8 hundreds, 1 ten, 6 ones, into 4 equal piles, we can put 2 hundreds and 4 ones in each pile. A zero, however, must be written in the ten's place between the hundreds and the ones to keep the hundred's digit in the proper place.

The illustrations thus far have been given to indicate how each process with whole numbers may be shown to be a part of a unified system of ideas. Each phase of a process grows in complexity but the same general principles apply to all phases. When pupils grasp the basic idea of grouping in tens or powers of ten, they should not experience greater difficulty in carrying from tens to hundreds than in carrying from ones to tens. They also have the background of understanding for carrying a number other than one when column addition and multiplication are taught.

The fundamental processes with whole numbers may be extended meaningfully. In the illustrations which have been offered, place value relationships have been limited to those between ones and tens and hundreds. For each illustration it has been suggested that the introduction of a new process should be in the form of a record of what transpires when concrete materials grouped as hundreds, tens, and ones are manipulated for specific purposes. One may logically ask whether or not this procedure should be continued when pupils first deal with numbers containing thousands. In other words, do children reach a point at which place value has been generalized to the degree that concrete materials are no longer needed?

In that number meaning plays a vital part in a program of arithmetic instruction, which is based upon a knowledge of place value, it would seem that first children must form concepts of larger numbers if they are to operate with them. Thus the problem of determining when children may discontinue the use of concrete materials in the learning of new processes raises a double question. Specifically it is—can pupils extend and enlarge the meanings of numbers from tens and hundreds to thousands, and can they manage numbers with thousands after operating with tens and hundreds without direct reference to concrete objects? There is no valid answer to this question.

If the descriptions of introductory lessons offered by Wheat represent his conclusions on the matter, it would seem that he does not advocate the grouping of concrete objects beyond that of demonstrating the number of tens in one hundred. In his discussions he suggests: "It should not be necessary to explain the meaning of every set of symbols from 20 to 100 or to demonstrate objectively every idea represented by them. It is assumed that the pupils, through various demonstrations and activities previously described, have already developed some fairly definite notions of the group of ten and of thinking numbers beyond ten in relation to ten."[13]

The same opinion is voiced in another statement by Wheat: "Of necessity, the pupils must rely less and less upon experiences with the concrete and depend more and more upon the system of dealing with tens just like units."[14] Wheat, then, would extend the processes of addition, subtraction, multiplication, and division to larger numbers by helping children realize that the method of dealing with tens and ones applies also to larger units. Others who have attempted to apply this meaningful approach to the computational phase of arithmetic as it relates to whole numbers agree with this viewpoint.

Whether all pupils are able to transfer the principles learned objectively about ones and tens to larger units is an open question. It is conceivable that many children of the ages at which larger numbers are first introduced may find it difficult to make the necessary transfers of ideas. For example, it may be necessary to employ objective materials for the purpose of leading pupils who have dealt with numbers through 100 meaningfully to perceive that 1,000 is not only 10 hundreds, 100 tens, but also 1,000 ones. The problem of transfer becomes even more serious when children find it necessary to learn how to operate with multipliers and divisors of two or more places.

Place value in long multiplication. In a simple problem such as 24 × 48 the multipliers 4 and 2 have the values of 4 ones and 2 tens. In multiplying by tens, is it sufficient for pupils merely to think, "tens are multiplied just as though they were ones," or should an effort be made to have children rationalize the operation by thinking, "2 tens times 48 are 96 tens, or 960"? Buswell, Brownell, and John[15] advocate the writing of the second partial product as 960 at first, thus emphasizing meaning during the early learning stages.

In preparation for two-place multiplication, Buswell, Brownell, and John,[16] Wheat,[17] and others, have children multiply numbers

[13] Wheat, *op. cit.*, p. 250.

[14] *Ibid.*, p. 350.

[15] Guy T. Buswell, W. A. Brownell, and Lenore John, *Daily Life Arithmetics*, Book One, p. 441. Ginn and Company, Boston. 1938.

[16] *Ibid.*, pp. 439-440.

[17] *Ibid.*, pp. 315-316.

by 1 ten, 2 tens, and the like. Likewise, by also using 1 hundred, 2 hundreds, 3 hundreds, etc., as single number multipliers, meaning may be given to three-place multiplication. Obviously there is need for research which will indicate the amount of experience children must have to perceive the place value principles as they apply to number meaning and to the fundamental processes of addition, subtraction, multiplication, and division. Unless children understand what they are doing, they perforce must learn mechanically. It should be noted that this approach to the more advanced forms of multiplication and division probably requires a mental maturity greater than children may possess when these topics usually are studied.

The meaningful approach to division by tens. The teaching of division by tens has long been a stumbling block for many teachers. Taught mechanically, it necessarily must be difficult for most children to learn because it requires a complete control of many intricate steps. Taught meaningfully, children are able to guide their thinking and as a consequence many of the confusing elements are eliminated. Taught meaningfully, division by tens is but an extension of division with one place divisors. This in itself simplifies the teaching of long division.

At an earlier point an explanation of division based upon place values was presented. The form of the explanation was:

$$\begin{array}{r} 26 \\ 6/\overline{156} \\ 12 \\ \hline 36 \\ 36 \\ \hline \end{array}$$

156 objects are to be placed into 6 equal piles. 156 contains 1 group of 100, 5 groups of 10, and 6 ones. There being an insufficient number of hundreds to divide into 6 groups, the 1 hundred must be changed to tens, etc.

There would be very little difference between the explanation of this problem and a division problem such as $20/\overline{180}$. The two-place divisor presents little that is new if the multiplication and division tables have been extended to include the tens numbers. Viewed as division by 2 tens the problem might be that of putting 18 tens into piles with 2 tens in each pile. This brings into play the multiplication combination 9 × 2 tens or 9 × 20. Very little difficulty is experienced in leading children to extend the multiplication combinations to include combinations such as 9 × 20, 8 × 40, 7 × 60, 5 × 50, 20 × 9, 40 × 8, 60 × 7, 50 × 5, etc., which are very useful in long division. They find use for these combinations in solving such problems as —"If a car which costs $180 is to be paid for in 20 equal monthly payments, what will be the amount of each payment?" "How long will it take to drive a truck 180 miles at an average rate of 20 miles per hour?"—and the like.

The purpose of the problem $20/\overline{180}$ might also be to divide $180 between 20 people. Obviously no person can receive any $10 bills, there being only 18 of them to divide. Hence $1 bills must be divided. If the problem were to divide $184 among 20 people, they would receive only $1 bills. The remainder 4 offers no difficulty because it indicates the number of $1 bills remaining after each of the 20 people had received the largest number possible. Division with one-place divisors taught meaningfully may give pupils an understanding of all the elements of the above problems.

Progress from problems such as the above to one like $21/\overline{92}$ represents an easy step. In this case the purpose may be to find the number of sheets of paper which might be given to each of 21 children from 9 packages of paper, with 10 sheets in a package, and 2 extra sheets. It is apparent that each pupil could not be given a whole package of 10 sheets, necessitating the breaking of the packages into single sheets. Pupils accustomed to rounding off numbers can be directed in the solution of this problem to think 90 divided by 20. If there were 90 sheets for 20 children, each could receive 4 sheets at the most. This represents the beginning of finding trial quotients. It offers an explanation for use of the ten's figure as a guide in the determination of the trial quotient. In a mechanical plan of instruction usually no explanation is given.

The progress might next be to a problem such as $34/\overline{110}$. Again, if 11 packages of paper of ten sheets each are to be divided among 34 children, 30 children (the round number of children) could receive only 3 sheets each because $4 \times 30 = 120$ and $3 \times 30 = 90$. The proof of this problem would be $3 \times 34 = 102$ and 8 more makes 110. This proof is not unlike that of a one-place divisor problem with a remainder.

The long division studies which have been reported do not provide data with which to judge the efficacy of the *meaning* method as applied to long division. In particular, finding trial quotients by the mechanical method requires the application of certain steps in a definite order. Finding trial quotients by the *meaning* method represents the further application of number ideas which have been acquired over a period of time. Children very early may learn to round off numbers and to find approximate answers. Thus finding quotients by first rounding off numbers is not something "extra" which adds to the complications arising in long division problems.

The point in this discussion is that the teaching of any part of a process of arithmetic by the meaning method cannot be evaluated on the basis of the description of that part. Meaningful teaching represents a building-up process in which old ideas work together in new situations; therefore what often appears to be complicated and difficult is in reality very simple for the pupils who possess the proper background of instruction.

The application of place values to decimal fractions. Emphasis upon the basic principles of the number system throughout the work with whole numbers will pay dividends in the work with decimal fractions. Since pupils already realize that each numeral indicates the size of a group by its position and indicates also the number of such groups, they will be able to apply these basic ideas to a study of decimal fractions.

In the number 111 the first 1 on the left = 100, the next 1 to the right = 10, and the 1 on the extreme right means 1 unit. Thus 10 is 1/10 of 100, and 1 is 1/10 of 10. When we wish to represent 1/10 of 1 unit, we can do it by using some sign to designate the end of the whole number series, such as a dot which also ends a sentence, and then write a 1 after the dot. Thus we have 111.1, and we call our dot a decimal point.

If we continue our series to the right of the decimal point we see that we can represent 1/10 of 1/10 or 1/100 and so on. Many teachers center these experiences in money values. In every section of the country children are familiar with dimes and pennies. There are certain areas in which mill tokens are in circulation, and there are speed and distance meters on cars.

Computations with decimal fractions are made possible because of the extension of the same basic principles of the number system as were developed with whole numbers. Tenths, hundredths, thousandths, etc., may be treated just as units are treated if 10 tenths are regrouped as 1 whole; 10 hundredths as 1/10; 10 thousandths as 1/100, etc.

The topic of decimal fractions is treated more fully in the next chapter by Wheat. The only purpose of bringing it into this discussion is to indicate briefly how the principles which apply to whole number meanings and operations provide the foundation for more advanced work in arithmetic.

Summary. No attempt has been made to outline in detail a curriculum for the lower grades. That would involve matters pertaining to the selection of content, social as well as mathematical, grade placement, and other problems of curriculum making which are treated elsewhere in this volume. The aim of the writer has been to indicate how arithmetic in the lower grades, as it relates to whole numbers, may be taught meaningfully as a closely related system of ideas. Intimations have been made of how even the learning of the number combinations may be made meaningful rather than mechanical through the use of comprehensive general ideas or generalizations.

In the program presented, place value is the key to all operations with numbers. The attention of pupils is focused upon place value when they first deal with numbers larger than ten. It serves to help them think many of the addition and subtraction combinations. All operations with numbers larger than ten are first understood from the point of view of place value. The degree to which this must be carried before children acquire basic principles which will enable them to deal with larger number situations is as yet an unsolved problem.

Special emphasis has been placed upon the need for concrete material in a meaningful program of arithmetic instruction. This is in keeping with the idea that children should first make number records of situations in which objects are manipulated before moving to described or abstract levels. In short, an effort has been made to indicate the nature of an arithmetic program which may be termed meaningful.

THE PROBLEM OF CLASSROOM METHOD

If we wish pupils to grow in their knowledge and understanding of arithmetic as a closely knit system, it is necessary to point our teaching in that direction. Observation seems to indicate that only a small percentage of pupils gain this basis of understanding unless teachers aim consciously for it.

Teaching which aims merely to tell or to show a pupil "how" to proceed step by step in the mastery of a process, or which merely provides a rule to be memorized and then applied, becomes purely mechanical. Pupils then have no challenge to reason or think out "why" dictated procedures and rules work. On the other hand, pupils who come to understand the reasons for various steps in a process before they are required to use the process have the ability to redevelop it for themselves if memory and habit fail. It is contended further that a type of practice which includes practice in thinking and practice in seeing relationships becomes a definite aid in the mastery of facts or in the perfecting of the steps in a process.

In a program of arithmetic teaching which emphasizes relationships and aims to show how certain fundamental ideas bind all the simple processes and extensions of processes into a consistent whole, pupils are encouraged to develop resourcefulness in their attack upon new processes. If given the opportunity, pupils will often be able to discover for themselves the next step in a process. However, if the teaching has been purely mechanical with no attempt to study the characteristics of the number system or to emphasize general modes of attack, it is not likely that pupils will develop this resourcefulness.

The foregoing statements in a sense deal with principles of method. If the principles which have been enunciated are observed in the classroom, the means which are employed to provide children with learning experiences necessarily must possess certain characteristics. The principles to which reference is made were given at the outset in this discussion. They are repeated here for purposes of reference.

(1) Children should become acquainted with numbers and use them extensively as records of quantitative experiences.

(2) Successful and efficient extension of number usage from the early experiences of making number records of quantitative situations can be facilitated by insight into the nature of the number system.

Translated into classroom procedure, the learning of something new in arithmetic will begin with a problem involving concrete objects. Children will either manipulate objects themselves as a means of solving the problem or observe manipulations made by another. Symbolic number records will be made of their observations or of the manipulatory activities.

In accordance with the second principle, the experiences of observation, manipulation, and recording will culminate with the making of deductions or the discovery of generalizations which represent insight into the nature of the number system. For example, the pupil who makes deductions about carrying, in two-place addition, which will enable him to carry in other examples, has gained further insight into the nature of the number system. A very important goal of meaningful teaching, then, is that of guiding children in their discovery of useful generalizations inherent in the number system.

Usually some direction or tuition is required from the teacher to carry children to the point of applying new generalizations successfully. For the sake of brevity the course of events from initial experience to final generalization and application are placed in a step sequence. Briefly the steps suggested are:

(1) Experiences of a quantitative nature with concrete objects.
(2) Making number records of the quantitative experiences.
(3) Analysis of number records.
(4) Moving from concrete manipulation as a source of number records to a level of description, thus providing practice and application.
(5) Working on the abstract level.
(6) Further extension and application into social uses.

This sequence may be followed in the teaching of any *new* idea or topic of arithmetic, whether it be number meaning, factual learning, or a process of arithmetic. In the chapter by Sauble the use of concrete materials in the teaching process is described in detail. Therefore only brief descriptions of teaching procedures which employ the sequence listed above will be offered at this point.

How addition doubles may be introduced. It is expected that the setting for all new learning will be in some situation of social, economic, or cultural significance. Whether the situation should be a part of a larger activity or a situation proposed by the teacher or the textbook involves matters beyond the scope of this chapter. For purposes of arithmetic instruction it is important that new learning should have its roots in experiences of social significance.

Usually the addition doubles taught in the early grades include

$$\begin{array}{rrrrrrrrr} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ +1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 \\ \hline 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \end{array}$$

In a meangingful program the goal is not to teach a certain number of specific combinations but rather to lead children to acquire generalizations which will enable them to think answers for many related combinations. Thus in the teaching of the doubles the goal may be to lead children to discover some interrelationship among the addition doubles which will make them more meaningful, as, for example, a double succeeding any given double combination is 2 more, and the one preceding is 2 less than the sum of the given double.

Experience setting. The problem at hand may be that of buying two articles each costing the same amount. How realistic it will be to a group of children will depend upon the ingenuity of the teacher. Obviously the experiences of the children will be made significant to them if articles are properly labeled as to price and coins with which to make payments are at hand. There are some children who have profited educationally from experiences in making purchases to the extent that they can begin with described situations. Account of differences in experience must be taken by each individual teacher.

In the teaching of the addition doubles there is value in considering the doubles in order. The purchases may begin with a purchase involving 2 + 2. Coins may be laid out for each article and the total found by counting if necessary. The purchase then should be recorded as $\begin{array}{r} 2\text{¢} \\ +2\text{¢} \\ \hline 4\text{¢} \end{array}$ Making purchases and laying out coins may be continued, leading to the number records: $\begin{array}{r} 3\text{¢} \\ 3\text{¢} \\ \hline 6\text{¢} \end{array}$ $\begin{array}{r} 4\text{¢} \\ 4\text{¢} \\ \hline 8\text{¢} \end{array}$ $\begin{array}{r} 5\text{¢}. \\ 5\text{¢} \\ \hline 10\text{¢} \end{array}$ These records may be reviewed in the order of the purchases. Following this the children may be challenged to make a record for the purchase of two articles each costing 6¢ without referring to the coins as a means of obtaining the sum. The child who can write $\begin{array}{r} 6\text{¢} \\ +6\text{¢} \\ \hline 12\text{¢} \end{array}$ and continue with $\begin{array}{r} 7\text{¢} \\ 7\text{¢} \\ \hline 14\text{¢} \end{array}$ $\begin{array}{r} 8\text{¢} \\ 8\text{¢} \\ \hline 16\text{¢} \end{array}$ $\begin{array}{r} 9\text{¢} \\ 9\text{¢} \\ \hline 18\text{¢} \end{array}$ $\begin{array}{r} 10\text{¢} \\ 10\text{¢} \\ \hline 20\text{¢} \end{array}$ and $\begin{array}{r} 11\text{¢} \\ 11\text{¢} \\ \hline 22\text{¢} \end{array}$ etc. has made an important discovery if he does not already know these combinations. The doors have been unlocked to many other combinations. Teaching would offer few problems if children discovered generalizations as readily as indicated in this account.

Tuition is required for children who do not perceive the unifying idea that each succeeding double is two more than the one preceding it and two less than the succeeding one. In this instance it may be necessary to repeat the experiences of laying out coins, finding sums, and making symbolic records of each purchase. It may even be necessary to ask children to demonstrate a combination such as $\begin{array}{r} 2 \\ +2 \\ \hline 4 \end{array}$ concretely and then change the concrete representation of $\begin{array}{r} 2 \\ +2 \\ \hline 4 \end{array}$ to show $\begin{array}{r} 3 \\ +3 \\ \hline 6 \end{array}$ and in turn $\begin{array}{r} 4 \\ +4 \\ \hline 8 \end{array}$ and $\begin{array}{r} 5 \\ +5. \\ \hline 10 \end{array}$ Furthe direction may be required to the extent of asking what is done to each number to produce the next higher combination. Obviously the discoveries of some children will be few and will come slowly.

Nevertheless pupils derive satisfaction from making discoveries no matter how small they may be.

Once a generalization has been perceived, it may be applied to described situations, thus providing practice and application. In this manner a double purpose is served; children are provided with opportunities to use a new generalization over and over again and at the same time the ability to apply new knowledge in social situations is developed.

Finally a point is reached at which the number combinations may be dealt with as abstract numbers. Some children reach this stage much more quickly than others. However, children who have had meaningful experiences with numbers during the learning process do not need to resort to guessing or finger counting when dealing with abstract combinations. They can think answers.

In the program under discussion there is a place for drill if it is conceived to be practice in thinking from lower to higher levels. The chapter by Buckingham deals with the drill phase of arithmetic. The writer is in accord with his point of view.

Generalizations, like other types of knowledge, are forgotten and must be rediscovered. It has already been suggested that it matters not whether children apply the particular generalizations with which sets of combinations were originally connected. Thinking about numbers becomes more important than recalling specific number facts.

In the brief account of teaching the doubles combinations the following steps appeared in the description:

(1) Concrete objects were manipulated for a definite and worthwhile purpose.
(2) Number records were made of the quantitative experiences.
(3) Provision was made for the discovery of a useful generalization.
(4) Practice and application were provided through described situations.
(5) Children finally dealt with number combinations on an abstract level.

Using tens in subtraction. When using tens in subtraction is introduced, it is assumed that pupils possess the concept of subtraction as it relates to "take-away" comparison and additive types of situations, that they know the simple subtraction facts, and that they have a working knowledge of place value. It is also expected that they can subtract both one-place and two-place numbers from two-place numbers in subtraction problems which do not require adjustments between the one's and ten's places of the minuend.

The procedure for the introduction of using tens in subtraction may follow the course suggested for the teaching of the doubles of addition. The need for subtraction with so-called "borrowing" may be centered in a social situation. Objects may be manipulated and number records made of the activity. Pupils may move to the making of records of described situations. This may be followed by further applications and practice on the abstract number problems.

It is important that the social situation should involve objects which are usually grouped as tens, hundreds, thousands, etc. Our paper currency lends itself admirably to the purpose at hand. Other objects such as sheets of paper, counters, tickets, and the like, may also serve the purposes of the teacher.

Suppose the social situation under consideration or the activity in which children are engaged required the payment of some such amount as $38 from $56. Then finding the difference between 56 and 38 becomes a problem. From past experiences children should be able to indicate symbolically the required subtraction in the form of $56
−38.

Meaningful teaching would, in this instance, require a supply of $1 and $10 bills prepared for classroom use. Six $1 and five $10 bills would be counted out. The problem then becomes one of removing eight $1 and three $10 bills from those representing $56. Not having a sufficient number of $1 bills it becomes necessary to change one $10 bill to $1 bills. By ingenious methods children can be led to discover this. The possession of sixteen $1 bills may be indicated by either placing a 1 in front of the 6 or by striking out the 6 and placing a 16 above it. Only four $10 bills remain, necessitating a record of this change, i.e., striking out the 5 and placing a 4 above it. Children with whom number meaning has been emphasized will know from previous experiences that four $10 and sixteen $1 bills is the same as five $10 and six $1 bills. The number record now is

$$\begin{array}{rr} \$\overset{4}{\cancel{5}} & \overset{16}{\cancel{6}} \\ -3 & 8 \\ \hline \end{array}$$

The completion of the problem is a simple matter. It is, however, suggested that children should apply their knowledge of the subtraction in giving the difference between 16 and 8 rather than obtaining it by counting. Counting is used for purposes of verification. Occasionally it becomes necessary for teachers following this plan of instruction to ask children who do not readily know that $16-8=8$ how they might think the answer. Replies are obtained such as: "$8+8=16$, therefore $16-8=8$"; "$16-6=10$ and $10-2=8$"; and "$8+2=10$ and $10+6=16$." This probably should not be necessary when this kind of subtraction is encountered.

When the difference of 18 has been found and recorded, it seems reasonable that the number record should be examined for the purpose of identifying the specific activities involved in manipulation of the bills. It is important that children recognize the 16 as the number of $1 bills at hand after one $10 bill had been changed to bills of the $1 denomination, the 4 as the number of $10 bills remaining after one has been changed, as well as the 18 as the one $10 and eight $1 bills remaining after $38 has been taken out of the $56.

The teacher must decide how many times children must deal directly with the actual bills before he proposes the finding of a difference without the aid of them. When children are ready for that step the procedure becomes one of challenging children to "show with figures what you would do if you actually counted out the bills." Computational arithmetic deals with records of quantitative experiences on the thought level. Individual differences among children require a repetition of manipulation, recording, and analysis in most classrooms. However, children who understand the new process may be given the opportunity of working on the thought level as soon as they are ready to do so. This is a matter related to teaching technique.

Mention has already been made of the problem of transfer from dealing with numbers which represent objects which are usually grouped as tens, hundreds, thousands, etc., to numbers which represent objects not commonly grouped in this manner. Proper attention to the development of number meaning throughout the early grades should carry children to the point where they will deal with all numbers decimally.

Attention is called to three significant points in the application of "meaning theory" of arithmetic instruction to the subtraction process. First, children do not use the term "borrow," which is entirely incorrect. Instead of "borrowing," they change a given number to another form. In the subtraction of 38 from 56, the 56 is changed to 40 and 16 by converting one of the tens to ones. Second, no rules are taught. They are derived by the pupils in the form of a description of the sequence of events which takes place when the bills are manipulated. Thus whenever rules are formulated, they are in terms of the experiences of individual children. When subtraction is taught mechanically children frequently borrow because the subtrahend number is larger than the minuend number of the same place value. Taught meaningfully they make necessary changes because it would be impossible to perform the operation concretely without doing so. Third, the pupils discover the methods employed in subtracting from minuends in which there are fewer ones than found in the subtrahend.

Problem solving. In a meaningful program of arithmetic instruction, problem-solving instruction does not assume an independent role. Instead it is intimately bound up with the whole teaching process. The experiences with concrete settings through which abilities are developed provide experiences in using number for purposes of quantitative thinking. Following this, the applications of these abilities to situations which are described rather than "present-to-sense," contributes to the development of problem-solving ability. The recurrence of the need for certain abilities provides the necessary review.

The important factor in this program is that of experience. In many classrooms children obtain correct answers by recalling a tool which has been previously connected with other problems. Usually key words in the problems suggest the cue for its solution.

However, if numbers serve at all times as records of what actually would transpire if the activities suggested by the described problem were performed, problem solving requires a background of experience. For example, children may mechanically find the average number of miles per hour, the speed of travel, by having learned the rule that the number of miles traveled is divided by the number of hours of travel. Placed upon a meaningful basis children must, among other things, appreciate what it means to travel a certain number of miles per hour, and that traveling at that rate for several consecutive hours adds a given number of miles per hour. It would seem that dramatization, verbal description, and illustration should play a larger part in the development of problem-solving ability than has been true in the past. In the chapter by Sauble the use of materials and devices for the purpose of providing experiences is discussed.

CONCLUSION

It has not been the purpose of the writer to present in this chapter an instructional program for the lower grades but rather to present a point of view. Illustrations have been offered to indicate how generalizations may be featured in the teaching of number meaning, number combinations, and operations with numbers. In short, an effort has been made to place before the reader the writer's conception of a program of arithmetic instruction for which McConnell has given the psychological principles. Several programs embodying these principles have been developed and are now in operation. Further experience with them coupled with much needed research will bring about many refinements. It seems safe to assert that in these programs many changes in grade placement will be made down as well as up, a larger supply of concrete materials than is now found in most classrooms will be used, new measures of arithmetical learning will have to be developed, new problems of individual differences will be raised, changes will be made in the methods of teaching, and above all there will be a need for a new type of teacher training.

Chapter V

A THEORY OF INSTRUCTION FOR THE MIDDLE GRADES

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*. . . Your teaching of arithmetic . . . may merely train your class in a number of processes which will let them pass an examination at the end of the term. That is "useful." It may also help them manage their savings accounts better or get a job on graduation. That is useful too—and this time without quotation marks. But if you can develop in them an understanding of number relations, if you can teach them to visualize distances and quantities, to appreciate imaginatively the meaning of "ten million" or of "one-thousandth of an inch," then you are training them culturally: they will forever after be more sensitive, more appreciative, more understanding, even though they may do no better on a formal examination. (From Henry W. Simon, "*Preface to Teaching,*" p. 34. Oxford University Press, New York, 1938.)*

ARITHMETIC is a system of ideas. It is not a collection of objects. It is not a set of signs. It is not a series of physical activities. Arithmetic is a system of ideas. Being ideas, arithmetic exists and grows only in the mind. It does not flourish in the world of things. It does not arise out of sensory impressions. It has nothing to do with the amount of chalk dust forty pupils can raise in a schoolroom in thirty minutes. Arithmetic exists and grows only in the mind. Being a system, arithmetic must be taught as a system. It is not an outgrowth of the individual's everyday experiences. It is not learned according as the interests or the whims of pupils may suggest. It is not anyone's personal discovery or invention. Arithmetic must be taught as a system.[1]

Arithmetic, as a responsibility of the school, seems thus to create a kind of paradoxical learning and teaching situation, to furnish two opposing sets of demands. On the one hand, arithmetic cannot be imposed upon the pupil, but must be developed by his own

[1] In this chapter, arithmetic is considered as a system to be acquired by a study of number relationships. Different points of view with respect to ways in which relationships may be used are developed in chapters by Brueckner, Sauble, Thiele, and others.—(*Editorial Board*)

individual responses and reactions. On the other hand, arithmetic cannot be left to individual caprice, but must be made the subject of explicit instruction. Putting the two apparently opposing sets of demands together, we seem to avoid our paradox. Thus the pupil learns arithmetic only as he builds up meanings for himself, and thus he learns arithmetic only as he builds up meanings that are consistent with the number system which society through its school impresses upon him. In other words, the pupil must be an active learner, but he must be taught; he must be subjected to instruction, but the instruction must be individualized.

The thesis which it is the purpose of this chapter to present is that the demands of the number system on the one hand and the interests and experiences of the individual pupil on the other are not necessarily incompatible. The learning and teaching paradox, to which reference has been made, will be presented in a few typical settings, or in a few of its typical forms; and effort will be made to show how the apparently conflicting demands of the paradox may be resolved into a more or less consistent basic theory of instruction. Special reference will be made, wherever possible, to the work of the middle grades; but, for the sake of clearness, and because deficiencies in earlier instruction frequently exist and need to be made up, illustration and application will necessarily be made to reach down at certain points into the earlier grades. The discussions will develop the following main topics: I. The Number System versus Pupil Experiences; II. The Continuity of the Course in Arithmetic; III. The Continuity of Ideas in Fractions; IV. Individualization of Instruction; and V. Teaching Methods of Thinking.

I. THE NUMBER SYSTEM VERSUS PUPIL EXPERIENCES

The number system and individual learning. At every point in his administration of the program in arithmetic the teacher is confronted by the contrast between what number thinking has come to be as a highly perfected science and the haphazard and unsequential methods by which it is frequently learned in the school. On the one hand, number as a science is systematic and consistent; on the other hand, number as a practical art is often a series of unrelated, mechanized, and rule-of-thumb procedures. The individual pupil is usually found to be far from any very complete acquisition of the number system which the race through long ages of trial has brought to its present stage of perfection.

The teacher tends to react to the contrast in one, sometimes both, of two ways. Perhaps the more common reaction is to neglect the number system and concentrate upon the individual pupil. The teacher notes the wide gap between the completeness of the number system and the highly incomplete accomplishment of the pupil, and, with apparent logic, is impressed with the enormous difficulty, if not impossibility, of successfully closing the gap. He turns away from any thought or suggestion of bringing the number thinking of the pupil into conformity with the number system, and chooses instead to select from the number system those items which seem to conform to the pupil, to his interests, experiences, and needs, or to what the teacher deems they are or ought to be. We shall consider presently the consequences of such reaction.

On the other hand, the teacher may give thought to the number system and how it has come to be. He notes how the system has come out of racial experience, and he is impressed, either through his own thinking or through the force of traditional practices in the schools, with the final success of the experiential learning of the race. He draws a parallel between racial development and a possible individual development of the pupil which he perhaps can foster.

It is easy thus to imagine a parallel between racial development and individual development, and to set up an argument that the conditions and situations that were instrumental in furthering racial progress should be duplicated in order to provide the motives and the settings for the progress of the individual. Indeed, it is always difficult for the present to break away from the influences of the past and to make adjustments to present conditions. But the attempt to duplicate the situations that confronted earlier societies in order to furnish the motives for learning in present society is a distortion both of modern situations and of the conception of the true motives for learning. Though the pupil must learn the number system the race has evolved—at least something of the system, the conditions that surround him are radically different. He has to learn in a short time what it took the race a long time to develop. He should be steered clear of the mistakes the race has made. He should not be allowed the long periods of experimentation with inadequate methods of thinking that the race through accident had to experience. He is surrounded from birth by a perfected number system in daily use by the older generation, and throughout his formative years, if not his whole lifetime, he may profit through such experience; whereas the race enjoyed no such advantage because it was at all points confronted with the task of creating a number system and bringing it to perfection.

The only item in the parallel is the number system itself. Through blind trials extending over long centuries the race has succeeded in creating and developing a number system; it is this same number system that the pupil should recreate and redevelop under the systematic guidance of the school. The race lacked guidance and consequently fell into many difficulties; the efforts of the pupil may be guided in the direction of the successful methods of attack that the race through trial and much error finally came to adopt.

The problem approach. Harking back to racial experience is the present-day theory that new topics, new procedures, and new lessons of any sort should be approached by way of a problem which is made to confront the pupil. In racial experience every number situation posed a problem and was a situation in which doubt was involved. How to determine the size of a group was once a problem, because then early peoples knew little about counting. How to divide a large group into smaller equal groups was once a problem, because then early peoples knew no systematic and certain method of division. They attacked their problems, to be sure, and arrived at solutions. Even then, the problems remained, because the solutions gained were uncertain ones. Finally, however, solutions were gained and the problems now no longer exist.

It frequently happens, nevertheless, that the school seeks to introduce the lessons it would teach as problems for the pupil to solve. Thus, a situation that requires addition in order to be grasped is presented as a means of leading the pupil to an understanding of addition; or, thus, the situation of borrowing and lending money which involves percentage is taken up for consideration as a means of making the pupil aware of the significance of percentage. The thought is that the pupil will feel a need for learning and have a motive for learning addition or percentage, as the case may be; and this, of course, before he has any realization of the part either addition or percentage plays in the situation before him. In short, whether consciously or not, the effort is made to reproduce the learning situations of earlier days with the thought, no doubt, of immediate results similar to those which in earlier days were finally brought to pass.

There are at least three obvious fallacies in such procedure. The first is the fallacy of time. The school life of the pupil is too short to permit the duplication of the slow progress the race made in the perfection of our number system. A second fallacy is the fallacy of assuming that the pupil's point of view with respect to surrounding situations may be inferred from the adult's point of view. The adult is perfectly clear that the situation of borrowing and lending requires for its complete understanding a knowledge of percentage; the pupil who has never been introduced to percentage does not know this. Moreover, the felt need for the learning of percentage is the teacher's rather than the pupil's. A third fallacy is the fallacy of paralleling the learning situations of the pupil with the learning situations of early peoples. The pupil may be a primitive individual but he does not live in a primitive society. Whether the school is ever justified in withholding learning for the sake of creating a problem, or in attempting to duplicate the necessarily incidental ways of learning of earlier days, is a matter of serious doubt.

Arithmetic from the pupil's experiences. The problem approach to the pupil's lessons in arithmetic frequently moves ahead from a method of presentation to a determiner of content. As a consequence, the teacher is confronted with the apparently conflicting demands of "mathematical" arithmetic—so-called, and "social" arithmetic—so-called. Here as elsewhere, the teacher does not feel himself to be entirely absolved from some measure of responsibility to the number system; while at the same time he feels impelled to seek in the pupil's experiences in various social situations the arithmetic the pupil shall learn. When he is reminded, either by his own forethought or by such experiments as that of Hanna,[2] for example, of the inadequacy of pupil contact with social situations as a means of determining and introducing the arithmetic he feels the pupil might learn, the teacher faces the problem of whether to make up the deficiency by what he considers inferior methods or to multiply the experiences set up for the pupil.

Perhaps the teacher should not look upon "social" arithmetic as a particular "kind" of arithmetic, distinct from other "kinds," such as "computational" and "informational." Perhaps he should look upon it, if at all, as a method, as a means to an end, as a mode of presentation rather than the thing to be presented. In such case, his point of view would be reversed. Instead of determining the arithmetic which the pupil should learn in terms of the pupil's experiences in various social situations, the teacher would find himself making use of the pupil's experiences as a mode of throwing light upon the arithmetic which he sets the pupil to learn. He would thus avoid the temptation of presenting a haphazard content; moreover, if he found such mode of enlightenment inadequate, he would not thereby be discouraged from varying his methods.

How and when the pupil's experiences in various social situations are to be utilized are perhaps the important issues. A choice of answers is offered in the following account of the activity of playing store that was set up in apparently identical ways in two third grade classrooms.

> As an exercise in social experience a "store" was set up in each classroom under the direction of the teacher. A counter, shelves, and goods to be "bought" and "sold" were provided in each, and the pupils took turns acting as clerks and customers. At this point the identity stops. In the one classroom, the storekeeping was undertaken

[2] P. R. Hanna and others, "Opportunities for the Use of Arithmetic in an Activity Program," *The Teaching of Arithmetic*, pp. 85-120. Tenth Yearbook of the National Council of Teachers of Mathematics, 1935.

to stimulate the learning of the simple additions and subtractions and a few of the simple multiplications. In the other, it was undertaken to provide one of several means of practicing the meaningful applications of the simple processes mentioned, *after* they had been learned and understood by the pupils. In the one classroom, the arithmetical operations were engaged in as a kind of distracting activity that was insisted upon by the teacher as necessary for the other activities of storekeeping which possessed meaning; and when the teacher lessened her insistence the other activities went along quite as well without the arithmetical operations as with them. In the other classroom, the arithmetical operations were not divorced by the pupils from the other activities, but were used to give them an exactness they otherwise did not possess. In both classrooms, there was manifested considerable interest in the experiences to be had, in the social situations presented, in the "social" arithmetic provided; but there was a difference in the character of the interest that was manifested.

The recall of experiences. Something should be said also about the kind of pupil experience that is of value. Again, the issue may be presented by means of an illustration.

A fifth grade teacher,[3] in the effort to develop a method of teaching the ideas of size and number in the common fractions to his pupils, found considerable discrepancy between the possession of the ideas and the ability to recognize their symbolic representation. In a preliminary test, he seemed to discover that the pupils knew very little about the relative sizes of the simple fractions. They made many blunders in recognizing the largest of three written representations, such as ½, ⅔, ¼, and ¾, 3/6, ⅜; and the smallest of three written representations, such as ⅔, ⅖, 2/2, and ¼, ½, ⅓; almost twice as many incorrect estimations as correct ones. However, when he began his instruction, he discovered at once that the concrete representations of the sizes of the common fractions, one-half of an apple, two-thirds of a circle, and so on, were not unfamiliar to the pupils. Indeed, when he gave the opportunity, most of the pupils were able to provide ample illustration from their own experiences, and gave clear evidence of the possession of the meanings of the fractions whose written representations had been confusing. The teacher found that he did not have to teach size at all at the outset, but rather the way of representing size. He found the idea of size present, but that it had to be recalled and concentrated upon. He, therefore, by suggestion and

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[3] T. L. Dotson, *The Development of Administration of a Method of Teaching the Ideas of Size and Number in the Topic of Common Fractions.* Master's problem on file in the office of the Graduate Council, West Virginia University, Morgantown. 1938. This and other similar studies are used for illustrative purposes in this chapter primarily because they are representative of the type of qualitative research into teaching procedures which teachers themselves can carry forward amid the busy activities of the classroom.

> otherwise, encouraged the recall of those experiences which served to bring to the fore the desired idea:
>
> *Teacher:* One of two hunters killed a deer. How do you think they made an equal division? Did one man take both front legs and the other man both hind legs?
>
> *Pupil:* No, that would give one man more than half. They would cut the deer straight down the middle following the backbone.
>
> *Teacher:* How do you know that would divide the deer into halves?
>
> *Pupil:* Because my father furnished the feed for a pig, and another man furnished the pig. They meant to divide the pig equally when it was killed. They took an axe and chopped right down the middle following the backbone.[4]

Another illustration of resort to a particular type of pupil experience as means of recalling ideas already possessed, of concentrating upon such ideas, and of giving them further development, is drawn from the work of a teacher in the sixth grade in introducing percentage.[5]

> The work was introduced through recall by the pupils, under the stimulation and guidance of the teacher, of experiences with per cents in connection with grades on their spelling test papers, with commission received by their classroom on a seed-selling project, with the payment of the 2 per cent sales tax, and so on. Later, their experiences in dealing with fractions and decimals were recalled. Throughout, the effort was made to head up familiar meanings toward a new form of expression.
>
> One of the pupils had memorized the relation between the expressions 1/4 and 25%, but persisted in representing other expressions by an incorrect shorthand method of his own: 3/4 equals 40%; 1/3 equals 30%; and so on.
>
> *Teacher:* If 1/4 of a dollar is 25 cents, what is 3/4 of a dollar?
>
> *Pupil:* 75 cents, because 75 cents is three times 25 cents.
>
> *Teacher:* Can you change 3/4 to a decimal?
>
> *Pupil:* Yes, divide 4 into 3. (The division was performed correctly.)
>
> *Teacher:* Can you change your answer to per cent?
>
> *Pupil:* Yes, .75 equals 75%.
>
> *Teacher:* Now change 1/3 to per cent.
>
> *Pupil:* 1/3 equals 30%.
>
> *Teacher:* Are you sure that is correct?
>
> *Pupil:* I can divide and see. (The pupil divided correctly, and corrected his error.)

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[4] *Ibid.*, p. 28.

[5] Myrl G. Rohrbough, *The Development and Administration of a Method of Introducing the Idea of Percent to Sixth-Grade Pupils*, pp. 27-31, 53-54. Master's problem on file in the office of the Graduate Council, West Virginia University, Morgantown. 1939.

It should be added to the presentation of the two illustrations which have just been offered that in each instance the meanings of the pupils were headed up through the recall of experiences toward a unity, which in a sense was a new meaning, a new idea for use both in discussion and in application; and that as the unity took shape the pupils went about their exercises with absorbing interest.

Number ideas in new situations. In the uses of the experiences which we have described, it should be clear that the intention has been to suggest the development of number ideas, and not merely computational activity which may or may not aid in such development. The preceding chapter has suggested activities which lead to generalizations relating to the meanings of addition and the other operations in earlier grades; the present discussion has suggested uses of experiences which aid in building up the meanings of the fraction and the per cent. The uses to which the latter number meanings can be put have not as yet been indicated.

The pupil experiences which to this point have been indicated in the illustrations are those which have become familiar, either through out-of-school contacts or through the instruction of the school. But the pupil must gain new experiences. The concept of "social" arithmetic suggests it. The nature of the world in which the pupil now lives and into which he will shortly be thrust as an individual having responsibilities requires it. What of such new experiences? What part do number ideas play in gaining them? Upon what shall the greater emphasis be placed—the number ideas or the new experiences? Here again the teacher is confronted by the apparently conflicting demands of "mathematical" arithmetic and "social" arithmetic.

Whatever may be the answers that come first to mind, upon one point at least we can be clear. The situations which the teacher sets up for the sake of pupil experience in arithmetic have all within them a number phase or a number element. Where emphasis shall be put depends upon the relative importance of the number phase or element. It is suggested here that the number element in number situations is the essential element, and that a subordination of the number element to others is comparable to the production of *Hamlet* with the part of the Prince of Denmark played in a minor role. Let us turn to the work of the upper grades for illustration.

In the upper grades the pupil is introduced to the personal and business situations which are common to the everyday activities of people—situations which are everywhere faced by adults and which as an adult he cannot avoid. He is taught about the lending and borrowing of money, and the interest which is charged and paid for the use of money. He is taught about buying and selling, and the profits and losses which are incident to such transactions. He is taught about saving and investing, and the returns which result therefrom. He is taught about insurance, and the obligations and benefits which belong thereto. He is taught about installment buying, and the necessary charges and costs thereof. He is taught about discount, commission, taxation, and other related situations, which are of immediate interest to older members of his family, if not already to him.

In all such personal and business situations, the pupil is required to give attention to a common factor, whether or not he recognizes it as common. The common factor is the factor of quantity, of size, of amount, of magnitude. Thus, he must learn not only that interest is paid, but also how much; not only that money is made and lost, but also how much; not only that an insurance policy requires a premium, but also how much; not only that the installment agreement of "a dollar down and a dollar a week" is a convenience with a price upon it, but also how much; in short, that the factor of "how much" is the factor of essential significance and central concern. Moreover, in all such personal and business situations, the quantitative side is relational in character; and it is this relational characteristic that the pupil must be aware of if he really gains a grasp of the situations. Thus, for example, the "dollar down and dollar a week" for three months on a ten-dollar purchase may appear as a small price to pay for the extra convenience provided, but the price assumes its proper proportions only when viewed as a relational amount.

The study of situations in the upper grades appears as a study of situations *per se.* It is in reality a series of exercises in the interpretation of wholes, exercises in which one whole amount is seen as a significant relation to another whole amount. It is in reality the application of the tools of number thinking, which the pupil is supposed to have acquired in earlier grades, to the simplest and most easily understood situations of the social and business life which surrounds him.

Practice in number thinking. If the teacher concludes from such considerations as the foregoing that the number element is a characteristic and significant element in all number situations, he will find in his conclusions the answer to the question of what to emphasize when number situations are called to the pupil's attention. He will also have a definite purpose in mind when he calls number situations to attention and when he invokes and provides experiences involving number thinking.

At the outset, before any given number idea has been acquired or developed or before any given type of number thinking has been engaged in by the pupil, the teacher will look ahead to the need of acquainting the pupil with certain personal, business, or social situations which contain the idea or require the thinking as a characteristic and significant element. The teacher will be aware of other important features in all such situations, features to which the pupil must give attention. He will be aware also of the aid the possession of the number idea or the ability in number thinking will contribute to introducing the other features and to making them clear, and of the difficulty, if not impossibility, of sensing the significance of such other features apart from their quantitative relations or when their quantitative relations are not understood. He will seek, therefore, to introduce the number element in advance.

Moreover, the teacher will be aware that the number element in situations is not a thing in a vacuum, as it were, whatever the importance it may seem to have in its own right. Accordingly, as was suggested in an earlier topic, the number element will be introduced in and clarified by situations which are already familiar. Later, as and when the number element becomes familiar, it will be put to use to lend familiarity to new and unfamiliar situations.

Let us illustrate the procedure just indicated by reference to the introduction, development, and application of percentage. As an earlier topic has indicated, percentage as a new language and as a new mode of expressing relations between quantities is introduced by suggesting past experiences which provide the necessary introductory meanings. As these meanings head up into a unity of meaning, development is provided by an abundance of illustrations, suggested and evoked by teacher and textbook. The illustrations are familiar situations which involve and require the thinking of two numbers together in terms of their percentage relationship. The illustrations are commonly called "problems," though they are not properly problems at all. Properly chosen, the illustrations do not leave the pupil in doubt, but throw further light upon an idea which already has been introduced. Sometimes the complete illustration is given, that is, "the problem is worked"; sometimes the illustration is given in part and the pupil is required to supply the missing elements, that is, "work the problem"; and sometimes the pupil is required to supply his own illustration, that is, "make up his own problem." In any case, the "problem" is the presentation of a familiar situation which illustrates the idea of percentage and requires the consideration of the percentage relationship between two quantities.

Finally, the pupil is prepared for introduction to new situations in which the percentage relations between quantities are involved. First of all, the pupil needs to be introduced to the general, qualitative features of the situations. Next, he needs to consider special cases, to view the quantitative features.

If the topic is interest, for example, the pupil must become informed of the needs of individuals and of businesses for the use of ready money, of the risks of lenders, of the obligations of borrowers, of the advantages the borrowers gain, of provisions for safety and security, of the similarity between using another's money and using another's property, of the methods of reimbursing the lender for the use of his money, and of like matters. The pupil needs to gain ideas of what interest is, why it is charged and paid, the situation that involves it, and the concerns of the people who are involved in it.

As and when the pupil gains the general ideas of interest, he can profit from consideration of special cases. Here again, he should have illustration after illustration; not of percentage merely, because the percentage relation has already become fairly clear, but of the situation of interest. Here again, he should have "problems" to "solve." The purpose is to throw light upon the situation of interest from various angles. To throw light, the illustrations must relate to matters which are as familiar as possible. The situation of interest is partly familiar through its general introduction. It is highly important that the number element be familiar. The value of the illustrations is determined in the main by the degree of familiarity of the number element.

The pupil as an active participant. Our discussion to this point has been devoted in the main to the resolving of the paradox which the apparently conflicting claims of the number system on the one hand and of pupil experiences on the other seem to present. We find that no paradox need exist, that the development of number ideas and the use and extension of experiences may be brought into relation each with the other. Or, to state the matter in another way, we find some excuse for a paradox in the thinking of the teacher, but no necessity for one in the developing ideas and experiences of the pupil. We need to give further and somewhat more pointed attention to the part the pupil plays in the learning of arithmetic. We shall do this, first, by indicating the continuity of the ideas which the pupil may gain in the whole course in arithmetic and particularly in the work of the middle grades, and, later, by considering certain general suggestions relating to the individualizing of instruction, and by canvassing the possibilities of teaching the pupil methods of self instruction.

II. THE CONTINUITY OF THE COURSE IN ARITHMETIC

The study of wholes. At the outset of his work in arithmetic the pupil studies small, easily apprehended groups, either incidentally or systematically. If the latter, he both learns systematic methods of study and employs such methods in his study of groups. He studies groups by counting them, by comparing one group with another, and by separating groups into component smaller groups and combining smaller groups into larger. He learns, by one means or another, to deal with the group of ten as a standard, and to combine and analyze groups in terms of the standard group. As he proceeds, he learns the language of number, both the oral and the written; and he learns to use the written language as an aid in dealing with groups as well as a means of recording the results of his thinking. Moreover, he develops certain general ideas of combination, the ideas of addition, subtraction, multiplication, and division. That is to say, he learns not only how to add, or divide, but also when to add, or divide, and why; or, to put the matter in another way, he acquires and develops meanings for the procedures of adding, subtracting, multiplying, and dividing. From first to last, if his studies are systematic, he not only handles groups, but also gives attention to what he is doing.

As the pupil moves along through the work of the primary grades, he learns to engage in what are called the more complex processes of the fundamental operations. This is only another way of saying that he gradually leaves the study of small groups which he can handle and study by direct means to engage in the study of groups which are too large to handle and which must be studied by indirect means. He learns, sometimes by rule-of-thumb methods, sometimes by deliberate attention to what he is required to do, to relate the combinations of the larger groups to the combinations which he earlier made of the smaller; and occasionally he is made consciously aware of the relationship, which is by way of the standard group of ten. Thus he learns, consciously and understandingly or otherwise, to group tens, tens of tens, and so on, in addition, subtraction, multiplication, and division, just as he grouped his ones at the outset. Throughout, he learns about whole numbers: first, those which are readily and directly grasped, and, later, those which need to be brought back in proper relationships to smaller and easily apprehended wholes.

The primary grades are the grades in which whole numbers are studied. In the primary grades the method of studying whole numbers is the method of combining and recombining whole numbers. Through the use of such method the pupil is carried at last to the point where he needs a new method of study if he is to continue to enlarge and clarify his ideas of wholes. The combining of wholes finally introduces such wholes as cannot be grasped through combinations alone. Then a new method of relating wholes must be learned if progress in thinking is to continue.

The study of parts. The new intellectual device to which reference has just been made is the idea of the part. The device is acquired through the study of parts, and the manner of employing it is learned through the study of the uses of parts. A double task thus confronts the pupil. He is placed in a situation which is comparable to that of the farmer when he bought his first tractor. The farmer first faced the problem of acquiring the tractor (of paying for it); he next faced the problem of learning how to run it.

At the outset parts are studied much as wholes were studied in earlier grades. First, each part is studied in isolation as a given sized division of a whole. Next, parts are analyzed into smaller related parts, and the smaller related parts are combined to form the original larger ones. Next, parts are compared as to their relative sizes, and, finally, they are combined and separated in additions, subtractions, multiplications, and divisions. Throughout, the language of parts is learned and used both to aid thinking and to state the results of thinking. In proper sequence, the various forms are learned and used: the language of common fractions, the language of decimals, and the language of per cents.

The study of parts, though it begins with everyday uses, quickly moves far beyond any everyday uses the pupil has or ever will have for parts. He engages in additions, subtractions, multiplications, and divisions of parts which are never employed by the common man outside the schoolroom. The purpose is to give a command of the idea of the part which transcends direct usage in ordinary everyday situations. The purpose is to give such a command as will enable the pupil to use the idea indirectly.

The part as a relational idea. Through the indirect uses of parts the pupil returns to the study of wholes. He learns to deal with wholes in a new way and by a new method. Now the pupil combines what he has learned of parts and of wholes, and employs parts as ideas and statements of relations between wholes. The new method of studying wholes does not substitute for the earlier methods, but supplements them, and thus provides larger and clearer ideas of wholes than were possible through the use of the earlier methods.

The pupil learns to deal with the part as the expression of a relationship between wholes through three kinds of exercises. In one exercise he learns to find an unknown number that is stated as a given part of a known number. In another exercise he learns to express a given number as a part of another number which is known to him. In a third exercise he learns how to find an unknown number when a given part of it is known. He engages in these exercises when he has studied common fractions, and again when he has studied decimals, and again when he has studied per cents.

In the middle grades the pupil studies parts. Through such study he learns, first, the direct uses of parts in ordinary everyday experiences and situations, and, next, the indirect uses of parts as expressions of relations between wholes. His study of parts thus proceeds to the study of wholes and parts. By means of the latter phase of his study the pupil is prepared for the study of the practical applications of wholes in succeeding grades.

The interpretation of wholes. When the pupil comes to the upper grades, it is assumed, often quite incorrectly to be sure, that he has learned how to think of one quantity as a part, or per cent, of another. In the upper grades, he is given opportunity to put what he is supposed to have learned to a use which can be seen to be of immediately practical importance. The fact that many pupils come to the upper grades not thus prepared merely serves to stress the responsibility which earlier grades are supposed to have met. On the surface it appears that the pupil is engaged merely in becoming acquainted with practical situations. Actually, however, he may be doing much more. He may be carrying forward the activities of dealing with parts as interpretations of wholes, of learning and using wholes, which had their beginnings in earlier grades.

The activity of interpreting wholes extends through the high school, though arithmetic as one of the common branches of study is discontinued at the level of the upper grades. In the high school every pupil must pursue what are known as the social studies. In such studies the pupil has his attention called to the institutions of society and to certain of the major problems with which organized society has to deal. He must study the activities of organized society in its attacks upon such problems—problems of public education, of poverty, of crime, of housing, of population, of social security, of industry, of capital and labor, and the like. Each set of activities or each problem is described in the pupil's textbooks and other reading materials. His task is the task of reading the descriptions.

The descriptions which the pupil is called upon to read are of two kinds, presented simultaneously. One description deals with the characteristics of a social institution, with the kind of social activity, with the nature of the problem. The other deals with their magnitude. One is a qualitative description; the other is quantitative. The one is printed in the language of the alphabet; the other is printed in the language of the Arabic numerals. Neither can be fully read and understood without the reading of the other. To read one and neglect the other results in a distorted view which misleads the thinking.

The numerals which appear in the descriptions of social institutions, activities, and problems must be more than pronounced. They must be interpreted. They do not impress themselves. They become impressive only as meanings are read into them and read out of them. The numerals represent large numbers, numbers which are far beyond direct apprehension, numbers which touch the individual only in the fractional degree in which he is a part of society, numbers which take on meanings only as they are brought down through many relations to the point where they touch individual lives more or less directly. To read such numerals, the pupil must bring to bear upon the activity all the resources of number interpretation which earlier grades can provide him. Unless he is well equipped, he is in constant danger of getting lost in the maze of relational thinking that is demanded for adequate interpretation.

III. THE CONTINUITY OF IDEAS IN FRACTIONS

The idea of number of parts. What has just been said about the possibility of a sequential and on-going single program of learning activity for the pupil throughout his whole course in arithmetic applies with equal force to the work that may be laid out for him in his study of fractions. His study of fractions can be made a program of developing ideas, such ideas as gain in clarity as his study proceeds and in turn render his study more and more unified and meaningful. To illustrate, reference will be made to the ideas of number and size of parts, to the new application of the decimal form of numeration, to the fraction as an expression of relationship, and to the so-called three kinds of problems.

The ideas of numbers which the pupil has developed in earlier grades have general applicability. This fact the pupil has recognized from the outset. For example, when he learned to count, he became aware that he could count anything. He counted the buttons on his shoes, the pictures on the wall, the plates on the table, the trees in the yard, and, with equal facility and certainty, the pieces (fractional parts) of an orange on his plate, or the pieces into which a pie, or a cake, or a melon had been cut. Similarly, when he learned the combinations, he became able to combine parts as well as wholes. Thus, for example, he could add the 3 pieces of orange he had had and the 5 pieces he was yet to have quite as well as he could add 3 oranges and 5 oranges.

The fractional parts which the pupil is thus able to count and to rearrange in various combinations are each regarded by him as a unity. To each he gives the same type of attention as is commonly given to the standard fractional parts of various measures. Though one may recall, if he is so disposed, that a quart is a fourth part of a gallon, he commonly regards each quart measure as a whole in and of itself. Similarly, he deals with minutes as unities, and with feet as unities, usually without considering them as fractional parts of the hour and the yard. Likewise, the pupil counts each piece of his orange as a whole piece;

to him the half-pint bottle of cream which he brings from the store is a whole bottle. As a consequence, the counting and combining of parts are at the outset identical with the counting and combining of wholes.

The school could, if it would, take advantage of the pupil's developed ability to count and to combine as one means of introducing him to his systematic study of fractions. While the idea of equality of sizes is being introduced, and is as yet not fully grasped, the pupil might well be directed to the counting of parts and to the combining of parts in additions, subtractions, multiplications, and divisions, treating the parts as he has heretofore treated them, that is, as unities. In such activities, the pupil would find himself on familiar ground while he is trying to gain another view that is as yet relatively unfamiliar. He would gain confidence in his ability to deal with fractions which otherwise are being explained in terms that temporarily at least seem to lessen his confidence. Moreover, his induction into the computations with fractions that are eventually required would thus be by easy stages.

The idea of size of parts. Just as the pupil's ideas of numbers may serve to connect his earlier study of wholes with his new study of parts, so may the idea of size become a unifying idea as his study of fractions proceeds. At the outset, the sizes of parts are called to attention. The half is introduced, explained, and illustrated as one of two *equal* parts, and the third as one of three *equal* parts. Moreover, the relations between the half and the fourth, the third and the sixth, the half and the third, and so on, are made clear by illustration. Attention to size is the first mental activity set up and directed in the introduction of fractions.

Somewhere along the line, however, the pupil usually fails to continue to hold his attention to size and to relative sizes, and at such point his difficulty begins. Many pupils in the school meet with much difficulty in dealing with the denominators of fractions. Such difficulty is with that feature of the fraction that both historically and in the experience of the pupil is the first to strike the attention. The first fractional numeration was the form of the unit fraction which concentrated attention on size; and the young child without benefit of schooling is never at a loss to distinguish between the sizes of two pieces of candy when he is offered a choice. But because teachers and textbooks either take an understanding of size for granted, or neglect to stress the importance of size, or mislead the pupil in his consideration of size, much confusion results.

The pupil comes to the study of fractions after learning to deal with numbers. Perhaps, therefore, his understanding of number in the study of parts may be taken for granted. But not so the feature of size. He must be taught to think exactly with regard to this feature. Very commonly the pupil is misled in his consideration of size by mistaken definitions forced upon him by teacher and textbook. It is the common practice to advance such definitions as the following: "The denominator shows the *number of parts* into which the thing has been divided." "The numerator shows the *number of parts* to be taken." In short, both denominator and numerator are said to show *number of parts,* which is half false and half true; and then, in order to indicate a distinction where none has been made, a misleading phrase, like "to be taken," or "that you have," etc., is added to the definition of the numerator to becloud its meaning. After such definitions, the teacher wonders why the pupil, who at the outset has no trouble distinguishing halves, thirds, fourths, etc., proceeds to add two-thirds and three-fourths as follows:

$$\frac{2}{3}+\frac{3}{4}=\frac{5}{7}$$

The pupil, having been confused about sizes, or having no systematic guidance in the recognition of sizes, or having been taught to treat them as numbers, knows very well that he can add 2 parts and 3 parts, and the result will be 5 parts. He proceeds so to add, as just indicated. Since the denominators in the addends possess a false meaning, he might just as well set down a false meaning (or any meaning) in the denominator in his answer.

The history of fractions reveals that the race made little if any progress in the development of ideas of fractions so long as size was emphasized at the expense of number, or so long as number was emphasized at the expense of size. The same revelation is to be noted in the case histories of pupils in the school. They make no progress in developing a working understanding of fractions so long as either distinguishing feature of the fraction is neglected in their thinking. Pupils must be led to study size as well as number. They must not be diverted from the study of size once such study is begun.

The decimal expression as an extension of decimal numeration. The characteristic feature of the Hindu-Arabic numeration, to which the pupil is introduced in the early grades, is the positional value which attaches to the numerals. In learning the system thereby represented, the pupil learns first to deal with groups to ten and next to deal with tens and multiples and powers of ten. He learns to use the numerals to represent numbers (of individuals to ten, and then of groups), and to use position to represent size (of groups). Number and size as coordinating ideas thus properly must be encountered, whether intelligently or not, before the pupil is introduced to his study of fractions. Here, again, we note the possibility of continuity in the ideas which the pupil may be led to develop. The continuity that is indicated may extend into the pupils' study of decimals.

If the pupil who is to be introduced to decimals has learned the decimal notation, that is, has come to an appreciation of the significance of its characteristic feature, and if, in addition, he has been kept aware of the ideas of number and size in his initial study of fractions, he will find in his study of decimals nothing new to learn. In decimals, he needs to deal with only two or three or four standard sized parts (tenths, hundredths, etc.), and these he may represent in the old, familiar way: the numerals are used to represent numbers of parts, and the positions in which they are placed are used to represent sizes.

The fraction as a relational idea. The use of the fraction as an indication of relationship between two numbers begins in the pupil's activities dealing with division of whole numbers. In division, the pupil is frequently required to find one of the equal parts of a quantity by dividing the quantity by the number of parts desired. The answer secured, which is the quotient of the division, is not an absolute result, but one that must be thought of in its relation to the number that has been divided.

The question asked at the outset, and usually repeated as part of the description of the quotient, serves to bring the quotient into relation with the original quantity. Thus, when the pupil is asked, "What is one half of 24?" he finds and gives the answer, "12 is one half of 24." Thus, he uses the expression, *one half,* to describe the relation 12 bears to 24.

A preceding discussion has indicated the later uses in the practical affairs of life of the fraction as a mode of stating one number in its relation to another. The point was impressed that the relational idea of the fraction, which has its beginnings in activities that precede the systematic study of fractions, continues as a useful idea after the systematic study of fractions has been left behind.

The three kinds of problems. Not merely before and following the pupil's study of fractions is to be located a possible emphasis upon the fraction as a relational idea. Much of the study of fractions is a study which is intended to make clear the family tree of numbers. Thus, the continuity remains unbroken from beginning to end.

In the study of fractions, the pupil encounters what we have termed the "three kinds of problems." He meets the situation that requires him to find the part of a number; a second situation that requires him to find what part one number is of another number; and a third situation that requires him to find a number when a part of it is given. In decimals, the pupil meets the same three kinds of situations; again in percentage he meets the same three kinds. If no attempt is made to have the pupil make comparisons, he may remember the situations (if he does succeed in holding them all in mind) as "nine kinds of problems." On the other hand, if comparisons and relations are emphasized from the beginning, he will come to understand them as the "three kinds of problems"; and moreover he will understand their relations and distinctions. Studying the "three kinds of problems" in fractions helps in the further study of the same three kinds in decimals and still more in their study later in percentage. As the pupil moves from one chapter in his arithmetic to the next, he is aided in handling the new chapter on a higher level of understanding by the use of what he learned about the same situations in the preceding chapter. He approaches the study of the new chapter with some ideas about the new chapter already formed and ready for use. A general method of attack is thus gradually developed.

Moreover, in the later stages of arithmetic, the pupil is required to study what are called the "applications of percentage" —interest, savings, investments, gain and loss, cost and selling price, insurance, taxes, etc. Each topic is new and unknown before it is studied. Each is in some degree a separate topic. Each may continue to appear as something entirely new and different. But if the pupil approaches the study of each topic with an understanding of the "three kinds of problems" and of the relations and distinctions between them, he may be led to view the various apparently different topics, like interest, savings, and investments, each as further illustration of the same "three kinds of problems" he already knows about. Thus, in the study of interest, the pupil will learn about certain business practices previously unknown to him, and, what is more to the point, he will at the same time gain a better understanding of the old and familiar "three kinds of problems." Thus his general ideas will develop along with the gaining of new information, and thus his preparation for the study of succeeding topics will be strengthened. Such a procedure in studying brings into a single, unified scheme of thinking or method of attack what otherwise might easily be a dozen separate, distinct, and unrelated "applications" of percentage.

IV. INDIVIDUALIZATION OF INSTRUCTION

The meaning of individualization. The continuity of ideas which runs through the course in arithmetic may become the pupil's continuity. This is to say that the pupil himself may make the ideas and the thread of their relations his own possessions, or in other words, that the instruction administered must have an individual bearing.

Individualization of instruction has a double meaning. The following excerpts from the report of an elementary school principal on the methods pursued to develop number ideas among the pupils of her school give illustration of the double meaning of individualization.[6]

> (1) *Individual Observation.* Since meaning and understanding are the main goals, the teachers should not depend too greatly on the results of written and oral tests as evidence of progress in meaning and understanding. Analysis of the daily oral and written work will aid the teacher in locating weaknesses and indicate some of the remedial work required; in addition to this, however, the teacher should make careful observations of the individual pupil while he is doing the work in arithmetic to find out his methods and how he gets his answers. If there are signs that the pupil is developing poor habits of working or thinking, proper remedial methods may be promptly applied. An instance of this observation and alertness on the part of the teacher is in connection with a girl in the fifth grade who was having difficulty in column addition; occasionally the answers she obtained were correct, but generally they were not. From observation and questioning the teacher discovered that the child always "carried" the larger digit; if the units column totaled 29, for example, she would write down the 2 and carry the 9, because, as she put it, "The tens are larger than the units." Special attention has been given to this child to correct this error, and eliminate the habit.
>
> (2) You will note, throughout the activities in these divisions, the unity and simplicity of the method. The children, under the guidance of the teacher, are discovering the number facts for themselves. Naturally practice must be given; the number facts must be rediscovered and reverified again and again, the grouping and the results repeated over and over until mastery is acquired. But the children acquire that mastery through practice, and, if they do not remember, they can always return to the group and rediscover and reverify the results. As an example of this, in one of our First Grades, a little chap who had developed his ideas through groups, noticed that his companion was in difficulty about the results of an arrangement. "Go back to your group," he said. "Here, I'll show you," and took the material and arranged it from the beginning, sturdily explaining that—"when you take two blocks away from the group, you have three blocks left, and when you put them together again you have the same five blocks."

The first of the foregoing quoted paragraphs illustrates the common and generally accepted meaning of individualization, namely, the application of remedial instruction according to the special deficiencies and special needs of individual pupils. The second of the paragraphs illustrates individualization in its broader and fuller meaning, namely, the stimulation and direction of the learning activities of pupils to the end that each pupil will develop ideas, meanings, understandings. Individualization may or may not be individual instruction. It may occur in the case of one pupil or in the case of each of several pupils in a group. In any case individualization implies that the pupil is made mentally alert to the essentials of the situation before him through the ideas and meanings which he develops and which thus become peculiarly his own. One may observe the process especially well in that type of instruction in the vocabulary of arithmetic in which the pupils learn certain words, not as mere words, but as names for ideas which already have come into their possession.

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[6] Ethel L. Nicholson, "Studying Groups as an Approach to the Development of Number Ideas in the Primary Grades," *University of Pennsylvania Bulletin*, 39: 296-297, June 26, 1939.

Learning as an individual matter. Let us take for example the acquisition of the word "division." In the midst of their study of a group of eight, let us say, the teacher raises the question, "How many twos are there in eight?" No one knows, of course, and no one knows how to find out. The teacher sets down before the pupils a group of eight:

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or takes charge of the group of eight on a pupil's desk, and demonstrates the arrangement into groups of twos:

●● ●● ●● ●●

The teacher now counts and has each pupil count the twos: "One, two, three, four." "There are four twos in eight" is the answer to which each arrives. As the pupils catch on to what is implied in the question and what the question requires, they each take over the activity of rearranging the group of eight into twos. The question may be asked orally, as indicated above, or in writing thus: $2\overline{)8}$. In either case, each pupil finds the answer for himself and gives the answer he discovers. Next, similar questions are asked: "How many threes are there in twelve?" "How many twos are there in ten?" $3\overline{)12}$, $2\overline{)10}$, etc., and for each question each pupil discovers and gives his own answer: "Four threes are twelve," etc.

The point of the illustration thus far is that each pupil is learning the meaning of division. The exercises in division continue. With each exercise the meaning of division is added to, becomes clearer, becomes more a personal possession. Finally, at some convenient time the teacher supplies the name: "We call this *division*." "When you do that, you *divide*."

Similarly, the development of such ideas as "average" and "per cent" is an individual, personal process. First of all, the pupil gains the idea; and, finally, he is supplied the name. Thus, in the case of the former idea, each pupil is stimulated to give his attention to the suppositions: "If he had traveled the same distance each hour." "If everyone had been given the same number." "Supposing he had received the same amount for each evening's work," etc. When each pupil has found the correct answer to a number of such exercises as are indicated, and more especially, when he has given attention to the supposition which gives meaning to the answer, the teacher supplies the name: "This is called the *average*." "We call this distance (or number or amount) the average distance." By a similar procedure with percentage, attention of each pupil is directed in a special manner to one of the several parts, or fractions, with which he has already gained some familiarity, namely, the hundredth part. Through discussion and illustrations, which are both supplied by the teacher and drawn from each pupil's experiences, attention is directed to several of the more easily understood special uses of the hundredth part. Finally, as the idea grows, the name *per cent* is supplied.

Individualization and grade placement. With the suggestion that arithmetic is a system of ideas which the pupil himself must be led to develop, belongs its corollary that arithmetic can be learned only as the conditions of maturation of the pupil provide. In recent years many have become greatly exercised over this matter of maturation, and certain ones have attempted to solve the issue by an external, predetermined scheme of grade placement. Their latest solution is that of meeting the difficulties of instruction in a given grade by passing along the difficulties to the teachers in succeeding grades. One may call to mind the fact that many teachers had been aware of this device of solving their problems long before the Committee of Seven made its original report.

It is the suggestion in this discussion that it is the pupil to whom we should turn for answer to the problem of maturation and grade placement of topics. Each pupil must develop the ideas of arithmetic for himself through methods of thinking which he may learn and use. As a consequence, the ideas in arithmetic develop no more slowly or rapidly than the pupil develops them. To illustrate, the pupils who are ready to learn the meaning and the name of the process of division in the manner which the preceding section has indicated are those who already have been engaged in an intelligent handling of groups. Similarly, the pupils in the experiment, to which reference will presently be made, will be found to have been ready to undertake the learning activities involved only when their teacher got them ready through preparatory exercises. And similarly, as an earlier section has pointed out, the pupils in the later grades who are ready to study the practical situations of life in which the idea of percentage is a characteristic element are those pupils who through their earlier training have come into possession of a working understanding of the percentage relation between quantities.

In what grade shall the pupil be taught percentage, or in what grade shall he be taught division by two-place numbers? The answers cannot be given in advance. The pupil may or may not be ready for the one in the seventh grade or for the other in the fifth. Everything depends upon his previous preparation, upon the extent to which he has been led to develop ideas of arithmetic and to use the methods of thinking in arithmetic in earlier years. There are many pupils in the seventh grade wholly unprepared to give attention to percentage, and there are many pupils in the fifth grade wholly unprepared to undertake division by two-place numbers. On the other hand, many sixth grade pupils can and do develop an understanding of percentage and many fourth grade pupils divide by two-place numbers with good understanding of what they are doing. The answer to the questions in either case is that the pupil is ready when he is ready. In other words, it is the pupil's use of the methods of thinking which he may be taught and required to use which sets the pace of his learning. This is only another way of saying that if we teach arithmetic according to its nature and peculiarities, we shall not need to worry about the issues of maturation and grade placement of topics; such issues will solve themselves.

If the issues require answers, they require answers which are relative, not absolute. The issues are much the same as that which could be raised about one's readiness to take any given step in going upstairs. When is one ready to take the sixth step, for example? The answer cannot be at eight in the morning or at four in the afternoon. He is ready to take the sixth step only when he has mounted without undue haste and with sufficient care to the fifth step of the stairs. In this connection Dantzig's story may be repeated.

Dantzig[7] tells the story of a German merchant of the fifteenth century who, desiring to give his son an advanced commercial education, appealed to a university professor for advice as to where he should send his son for training. The reply was that if the mathematical curriculum of the young man was to be confined to addition and subtraction, he could obtain such instruction in a German university; but if instruction was desired in the difficult arts of multiplication and division, he would have to go to the universities in Italy for such advanced training. (Fortunately, the grade placement of the four fundamentals did not remain at the university level. Fortunately, no fiat of grade placement kept them there.)

V. TEACHING METHODS OF THINKING

The direct versus the indirect method of instruction. The foregoing paragraphs may be summarized in the statement that the problem of grade placement is the problem of establishing appropriate sequence among the learning activities of pupils. The sequence, as has been implied, is internal in the developing understanding of the pupil, not merely external in the program and plans of the teacher. The sequence, if this is built up, turns out to be more than a mere arrangement of learning activities, primarily a means of guiding them. How internal controls of learning activities may be provided the pupil may very properly engage our attention at this point. Canvassing the possibilities of teaching pupils methods of self-instruction may help to reveal the part the pupil plays in the learning of arithmetic.

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[7] Tobias Dantzig, *Number, the Language of Science,* p. 26. The Macmillan Company, New York. 1930.

The learning and teaching paradox, to which reference has been made from time to time, confronts the thoughtful teacher as what may be called the basic problem of instruction. The basic problem of instruction is the question whether to teach the subject directly to pupils or to teach pupils methods of learning the subject so they can teach it, or much of it, to themselves. Shall the literature teacher teach poetry to his pupils, or shall he teach his pupils how to read poetry? Shall the science teacher teach the products of scientific thinking, or shall he teach his pupils the methods of thinking of science? Shall the arithmetic teacher teach arithmetic, or shall he teach his pupils the ways by which they may build up the ideas of arithmetic for themselves, and, as he does the latter, stimulate his pupils to use the methods to teach to themselves what arithmetic they learn?

The thoughtful teacher must weigh the merits of the direct and the indirect methods of instruction, as they are here defined; indeed, he must make a choice between them. He may not have to choose one method to the neglect of the other, for, as it may readily be seen, the two methods are not mutually exclusive. The literature teacher, for example, must teach some poetry as a means of teaching his pupils how to read poetry, and may very well teach some reading while he is engaged in teaching poetry. Nonetheless, the teacher must choose between the two methods, determining, if nothing more, which method he will emphasize.

At first glance it would seem that the direct method is the better method in the case of arithmetic, at least in the case of reaching the objective goals of arithmetic. The objective goals of arithmetic teaching which are commonly set up as the true goals, and which give the appearance of being the true goals, are no doubt reached more quickly and apparently more surely by the direct approach. But however much and whatever arithmetic pupils do learn under the tuition of the direct method, they learn something else also, something subjective, which is not commonly revealed by the tests and examinations at the end of the term. While they learn arithmetic, or such arithmetic as may be taught by the direct method, they learn dependence upon the teacher. On the other hand, by the indirect method of learning and using methods of self-instruction, pupils, though they may reach fewer objective goals or reach them less quickly, likewise learn something else also, something subjective, which is not commonly revealed by tests and examinations. While they learn arithmetic, such arithmetic as they teach to themselves, though under the constant stimulus of the teacher, they learn self-reliance. It is a sad commentary on the methods of teaching arithmetic in common use that the farther most pupils progress in the subject the more dependent upon the teacher and textbook they become. Conversely, it is a surprise and pleasure to behold a pupil who, by accident or in spite of the instruction he has been receiving, has discovered a way of explaining to himself the arithmetic he is learning, and has thus learned to rely more and more upon his own developing ability to understand. The thrill is almost as great as the thrill which such a pupil himself experiences!

Whatever may be the answer to the basic problem of instruction in other fields, in the field of arithmetic there appears to be only one answer. Arithmetic is a system of ideas. Arithmetic exists and grows for the learner only in the mind of the learner. Arithmetic must be learned in understood sequences or relations. The answer to the problem is that the only real way for pupils to learn arithmetic is to learn and use the methods of thinking which slowly and gradually cause the ideas of arithmetic to grow up in their minds.

Self-instruction a possibility. The objection may be raised that it may be very well to suggest methods of self-instruction in the later grades of the school, but that such methods are beyond the abilities of pupils in the elementary school, especially in such a difficult subject as arithmetic. It may be objected that in the earlier grades especially pupils are and must remain dependent, that they can take only such arithmetic as is given to them, and that only in homeopathic doses. To such objection it may be replied that even the beginner in the first grade does not have to be told that two and three are five, for example, *provided he has been taught to count as a means of finding out for himself the size of a group.* He can count out two, then three, and next he can put the two groups into one and count the five. Such a pupil is not taught that two and three are five. His teacher does not tell him. His teacher asks him what the answer is, and he finds it for himself. All that is required is that he be taught counting as a method of thinking and directed in the proper use of such method. Using such method and other somewhat similar methods, such as have been described in the preceding chapter, pupils in the first grade can find the answers, that is, teach themselves the answers, to forty-five additions and forty-five subtractions. Using such methods, they can, if their teacher desires, interpret simple questions in division and discover the multiplication answers.

Moreover, to carry the illustration further, pupils do not need to be taught the thirty-six more difficult additions. Instead, they can be taught a single method of grouping into a ten and so many more, and then they can use this method to teach themselves the thirty-six additions. They do not need to be taught the thirty-six corresponding subtractions as thirty-six separate bits of information to be learned. Instead, they can be taught a single method of subtracting from the ten, and then they can use this method to teach themselves these thirty-six subtractions. They do not need to be taught the simple multiplications, because they can work them out from the actual procedure of dividing larger groups into smaller equal groups and counting the groups. They do not need to be taught the more difficult multiplications and divisions. Instead, they can be taught the single method of transforming the groups to nine into groups of ten and counting the tens. They can be taught the simple procedure of reversing the method to determine the related divisions. Then they can use this method and this procedure to teach the multiplications and divisions to themselves. They do not need to be taught the complexities of the more difficult operations in the fundamentals. Instead, they can be taught the special significance of *ten,* and it special *position,* and the method of dealing with tens just as though they were ones. Just as soon as the pupil begins to be conscious of the fact that he can and does deal with tens, as he puts it, "just like ones," the complexities of the difficult operations are reduced to a single simplicity. He is in possession of the single key which he himself can use to unlock each successive door as he comes to it. Besides, as he moves from door to door, he gains greater facility and greater confidence in his use of the key. Finally, the pupils do not have to be taught how to solve problems in arithmetic. Instead, they can be taught the methods of work and the methods of thinking things out for themselves, such as an earlier section has indicated, and their problems turn out to be illustrations of number relations and of practical situations which they already understand in part and are engaged in examining.

What has just been said is that arithmetic is not an encyclopedia of facts and answers which the pupil is given and which he stores away for use, but instead a systematic method of thinking out number relations which, when employed, provides him with the facts and answers. What has just been said is that this systematic method of thinking is mastered only through use, and that it is the function of the teacher merely to introduce pupils to the method and to direct them in its proper use. From that point on, at every stage of his progress, the pupil is his own best teacher.

A third grade experiment in self-instruction. At this point evidence will be brought to bear upon the proposition that pupils can be trained in methods of self-instruction. Reference has been made elsewhere[8] to successful efforts to provide such training:

> *Pupils can learn self-reliance while they are learning arithmetic.* L. A. Gump taught two groups of second grade pupils a method of discovering for themselves the answers to the thirty-six more difficult additions, and P. G. Michael taught a third grade group a method of self-instruction in the thirty-nine more difficult multiplications. The pupils were successful in learning and in using the methods of self-instruction taught them. Every one of the forty-four pupils in the second grade groups succeeded in learning the thirty-six additions in nine trials on the average per pupil per combination. Every one of the forty pupils in the third grade group succeeded in learning the thirty-nine multiplications in twenty-one trials on the average per pupil per combination. Moreover, all the pupils concerned developed a degree of personal pride in their ability to depend upon themselves.

During the school year, 1938-1939, M. V. Givens[9] repeated and extended Mr. Michael's study in the case of fifty-four third grade pupils in two elementary schools in Marion County, West Virginia. Mr. Givens taught his pupils a method of finding the answers to the thirty-nine multiplications whose products are twenty-one to eighty-one. The method was that of grouping into tens. Thus, in the multiplication, *four nines,* the process taught was to group the nines into tens and to count the tens. The procedure in thinking was: "Four nines are three tens and six. Four nines are thirty-six." By illustration and explanation the point was made that the multiplication, *four nines,* asks, "Four nines are how many tens?"

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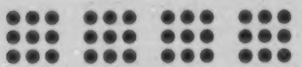
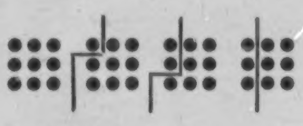
[8] H. G. Wheat, "Qualitative Analysis of Certain Imponderable Outcomes of Arithmetic Teaching." Abstract in *Research on the Foundations of American Education*, pp. 15-16. American Educational Research Association, Washington, D. C. 1939.

[9] M. V. Givens, *Using a Method of Self-Instruction in the Third Grade in Learning the Thirty-nine More Difficult Multiplications.* Masters' thesis on file in the office of the Graduate Council, West Virginia University, Morgantown. 1939.





Following instruction in the method, the pupils were given practice in using the method to determine each for himself the answers to as many of the thirty-nine multiplications as he had not as yet learned. The practice was individual. Each multiplication was indicated on the obverse of a small card, and on the reverse the grouping into tens was shown by means of dots. The pupil looked at the indicated multiplication on the front of the card and, if he could not give the answer, he looked at the grouping on the back and determined the answer, as indicated above, and gave it thus: "Four nines are three tens and six. Four nines are thirty-six." This he did once each day with each multiplication, each on its card in a pack, until he had exhausted the pack. Each day the pupil went through the cards of the pack until he had mastered all the multiplications. The criterion of mastery was the correct answer without looking at the back of the card on four successive days. Each answer to be counted had to be given in the time the teacher took to count *one, two, three, four* to himself. A daily record was kept of each pupil's answers.

The general character of the responses of the pupils is described by Mr. Givens, as follows:

> As the instructor worked with each pupil individually, it was interesting to note the development of two kinds of competition. First, the pupil was eager to surpass his record for the preceding practice. Invariably he would ask, "How many did I get right?" When told, he would compare his achievement with that of the day before, and if noticeable improvement had been made he would express satisfaction. The second kind of competition was that between individuals and the group. Often someone would ask, "Who is ahead?" Another question frequently asked of other pupils was, "How many multiplications do you know?"
>
> The attitude of the pupils in general toward the work was exceptionally good. This was shown by their eagerness to participate in the learning activity. No urging was necessary, for the pupils worked diligently at their assignment.

The general reactions of the pupils to the method of self-instruction are indicated by some of their remarks, such as the following:

"No one has to tell me the answer. I can find out for myself."

"The fives are easy to change to tens."

"You have more tens when the numbers (the factors) are big."

"Some of them are hard, but you don't have to tell me. I can count the tens and ones."

"If I forget, I can find out on the back."

"The answers are in dots. They help me to remember."

"I wish I could take these cards home. I'd learn them real quick."

"These are so easy I can learn them by myself."

. . . .

Certain of the author's conclusions are indicated thus:

The ninety-four pupils in the two groups required a total of seventy thousand three hundred eighty-six (70,386) responses to learn the thirty-nine more difficult multiplications, or an average of approximately nineteen and two-tenths (19.2) responses per multiplication. However, on the average, approximately thirty and four-tenths practice periods, during each of which all thirty-nine multiplications were studied, were required to learn satisfactorily the thirty-nine multiplications. Consequently during each of these practice periods an average of approximately one and three-tenths multiplications were learned.

The greatest number of responses required by any pupil on the multiplications as a whole was one thousand four hundred eighty-nine (1,489), and the least number of responses needed by any pupil was two hundred thirty-seven (237). The least number of practices required by any pupil on any multiplication was four, and the greatest number of practices required by any pupil on any multiplication was sixty-four. The average number of responses by individuals on the multiplications as a whole range from six and one-tenth to thirty-eight and two-tenths.

While the disparity in the amounts of practice required by the individuals of the two groups is conceded, in no case does the amount of practice appear indefensible. Instead, it suggests the desirability of methods of learning which can be adapted to individual needs.

It is highly significant that the ninety-four pupils were successful in learning the method of self-instruction, and that by this method each pupil taught himself the thirty-nine more difficult multiplications. That the pupils not only learned the multiplications but also enjoyed the learning activity was manifested by their attitude toward the work.

. . . .

Many interesting analyses of the responses of pupils were made possible in the course of the experiment, and the experimenter arrived at a number of interesting observations. With all these we have no occasion here to deal. It will be sufficient merely to repeat a caution of the experimenter in a statement of his purpose. He pointed out that his purpose was not to demonstrate the superiority of his method of self-instruction over other methods, nor to offer it in substitution for methods of practice and drill. He merely wished to try out the method to discover if pupils could learn it successfully. He did observe, however, that after his pupils had used the method to discover and verify products, they tended to abandon the method as the products began to become habituated.

An experiment with retarded pupils. During the school year of 1939-40, W. P. Cunningham[10] used the method of self-instruction, just described, with 40 retarded pupils in grades four, five, and six. He reports that the pupils all learned the method without difficulty and used it with interest and success to teach and reteach each to himself the unlearned or forgotten multiplications.

Other illustrations. Following are other illustrations of instruction in methods of thinking.

Reference has been made to the efforts of a teacher in the fifth grade[11] to call to the attention of his pupils the significance of the idea of size in dealing with fractions. The result was that his pupils all became conscious of size in considering the expression of a fraction. How large is each part? Are the parts in this fraction larger or smaller than the parts in such and such other one? How much larger? How much smaller? Such questions as these became questions which needed answering. Finally, when the pupils moved ahead to the addition of fractions, they were aware of the importance of changing the fractions to be added to "parts of the same size." To illustrate, when their teacher suggested proceeding at once to the addition of ¾ and ⅔, he was called to a halt by the admonition, "We have to get them to the same size, because you can't add fractions unless they have a denominator of the same size."

In an attempt to develop a method of instructing his pupils in the distinctions between the "three kinds of problems" in fractions and decimals out of their successful and unsuccessful ways of responding to such problems, a teacher in the sixth grade[12] found that instruction in the computational processes left the pupils dependent upon him to make the decisions about what process to use in a given case. He sat down with each pupil and tried to encourage him to "think aloud."

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[10] W. P. Cunningham, *Teaching Retarded Pupils a Method of Self-Instruction in Multiplication.* Master's problem on file in the office of the Graduate Council, West Virginia University, Morgantown. 1940.

[11] Dotson, *op. cit.*, p. 42.

[12] Rex M. Smith, *Methods of Learning to Make Distinctions Between the Three Kinds of Problems in Fractions and Decimals in Sixth-Grade Arithmetic.* Master's problem on file in the office of the Graduate Council, West Virginia University, Morgantown. 1937.

Then, in terms of the way the pupil seemed to be thinking, the teacher undertook to point out and describe the distinctions. His pupils had no trouble in distinguishing the "second kind" of problem. Their difficulties in the main were two in number: They were uncertain about the divisor and dividend in the "second kind" of problem, and they had no ready means of distinguishing between the "first kind" and the "third kind" of problem. The distinctions taught were as follows:

1. When you are asked to find what part one number is of another, put the *number being asked about* in the numerator or the dividend. This is the "second kind" of problem.
2. When a part is given *of a number that is given,* multiply by the part. This is the "first kind" of problem.
3. When a part is given *of a number that is not given,* divide by the part. This is the "third kind" of problem.

Through individual and group instruction the pupils were made conscious of these distinctions, and they became still more conscious of them as they attempted to note distinctions. In their practice exercises, if the problem was the "second kind," they asked themselves, "What is the number being asked about?" and they looked for this number. If the problem was not the "second kind," they asked themselves, "Is the per cent of a number that *is* given, or of a number that *is not* given?" and they looked to see if the number in question was present or absent. They were not always successful. They made mistakes; that is, they made incorrect decisions. But thinking was present, even when their objective responses counted for nothing. Looking for distinctions was number thinking, when the effort was unsuccessful as well as when it was successful. Having in mind what to look for contributed to appreciation of discovery even when they needed help in making it. Over a period of eight weeks the pupils reduced their errors, as indicated by objective tests, 66 per cent. The teacher had cause for thinking that the consciousness of distinctions among his pupils had improved beyond what was indicated by their objective responses.

A somewhat similar study of learning activity was carried on by an elementary school principal[13] among sixty-nine pupils in his eighth grade. Here the effort was made to teach pupils how to look for and to make distinctions between the "three kinds of problems in percentage." The method of instruction was the same as was pursued by the sixth grade teacher in pointing out distinctions between the "three kinds of problems" in fractions and decimals, and the results were much the same.

In his preliminary testing at the outset of his study, the principal seemed to find among certain of his pupils evidence of an ability which he later discovered they did not possess. For example, he found that certain pupils solved correctly a number of examples and problems which required the finding of the per cent one number is of another, but were in confusion on other similar examples and problems. The test seemed to show considerable ability, but not mastery. Upon later inquiry, his pupils revealed the secret of their methods. Some followed the rule: "Always divide the smaller number by the larger," and others followed the rule: "Divide by the number that comes after *of*." The former group did not understand why their rule which produced the right answer most of the time did not produce the right answer all the time, and the latter group was always in confusion when the statement of a problem made too free a use of the distinguishing preposition. This group was disconcerted by the *of* in problems that dealt with flocks *of* chickens, barrels *of* apples, and bushels *of* potatoes. Among both groups, number thinking had lagged far behind its objective manifestations.

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[13] F. D. Robinson, *Learning to Make Distinctions Among the Three Types of Problems in Percentage.* Master's problem on file in the office of the Graduate Council, West Virginia University, Morgantown. 1939.

Following instruction in how to look for distinctions, the pupils were given practice in looking for and trying to make distinctions. As was the case with the pupils in the sixth grade whose efforts have just been described, they too were not always successful in noting distinctions. Nevertheless, their ideas of what they were to look for were sufficiently clear for them to recognize discovery when their teacher or their classmates had to give help in making it. Over a period of nine weeks they reduced their errors 73 per cent. Here again, there was cause for thinking that the consciousness of distinctions had improved beyond what was indicated in the objective responses which were readily measurable.

Distinguishing between what should be taught and what should be self-taught. Running through the discussions which have just been concluded is the suggestion of a paradox between the instruction of the teacher and the learning of the pupil. On the one hand, the instruction of the teacher may lead to dependence; on the other, the learning of the pupil may involve much wasteful effort and lead to confusion. But the teacher must instruct, and the pupil must be much more than a receptive learner. The systematic character of arithmetic imposes the former requirement, and its subjective character imposes the latter. Yet a paradox need not exist, for in the total make-up of arithmetic are methods of work, which may be learned most expeditiously through direct instruction, and the products of such methods, which result only from individual usage. The teacher's responsibility is to teach methods of work and guide pupils in their use of methods; the pupil's responsibility is to use the methods and thus to teach himself their products. The teacher's problem begins with the questions: What are methods? What are products?

Partial answers to these questions have already been given and others may be suggested for illustration. Thus, the teacher should teach counting, but the pupil may use counting to determine for himself the sizes of groups. The teacher should teach the analysis of groups, but the pupil may use analysis to teach himself the various subtractions and divisions. The teacher should teach the synthesis of groups, but the pupil may use synthesis to teach himself the various additions and multiplications. The teacher should teach the special groupings into tens, but the pupil may use these special groupings to teach himself step by step the more complex processes. The teacher should teach the comparison of parts, but the pupil may use comparison to teach himself the sizes of parts and the relations between them. The teacher should teach the distinctions between the so-called three cases of percentage, but the pupil may use these distinctions to teach himself the special use of the idea of per cent in a given situation. The teacher should teach methods of work, but the pupil may become his own best teacher through his use of the methods.

VI. IN SUMMARY: A THEORY OF INSTRUCTION

Number thinking is a part of the total experience possible for the individual in many of the situations of his daily living. He must bring the thinking to the situations; otherwise the situations are powerless to stimulate the thinking which gives them clarity and exactness. Number thinking, therefore, must be cultivated in advance.

Number thinking is introduced and carried along by reference to personal experiences already gained which give it meaning. Without meaning, the procedure is routine, and not thinking. Moreover, number thinking is not in isolated bits, however much one may be able to analyze it once he has learned to engage in it; it is rather a unity, a system, a series of sequential relations. It is a process which develops through ordered methods.

Teachers may teach the pupil methods; it is a subversion of the purposes of instruction to permit the pupil to develop haphazard, unproductive, and confusing methods in his own way. The pupil's responsibility is to learn the methods and to use them as means of developing his own number thinking. Teachers cannot do the pupil's thinking for him. The number thinking that the pupil learns, therefore, is in a very real sense the product of his own self-instruction.

Chapter VI

ARITHMETIC IN THE SENIOR HIGH SCHOOL

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I. A NEW EMPHASIS FOR ARITHMETIC

THE purpose of this paper is to explore the possibilities of teaching arithmetic in the senior high school. Some readers will at once recall that this is being done, and may wonder why time and space should be devoted to a discussion of the desirability of a practice which is already accepted. However, the procedure to be recommended is not so commonly followed as some may believe; furthermore, the fact that a proposed educational reform is already established practice in some schools does not argue against advocating the extension of such a reform. This should be continued until the proposed revision of procedure either has been adopted as good practice in a considerable number of school systems with a comprehensive, critical, scientific program of curriculum revision, or has been scientifically evaluated and subsequently rejected by an equally significant number of schools of the same type. It can be asserted with confidence that neither of these two procedures is descriptive of the situation with reference to the proposal herein advanced. This will become more apparent as the thesis is particularized, definitions are advanced, and practice is described in terms of the objectives which the proposed program is supposed to achieve.

More arithmetic. The reader may well be on his guard against what may seem to be just another proposal on the part of representatives of subject-matter interests to extend the beneficent influence of their subject by adding a course in that field to the already overcrowded high school curriculum. This charge is frequently made, and it is recognized that the proposal herein made invites this criticism. The situation should be recognized. In the present instance it may well be admitted that the merit of the proposal should be judged by independent curriculum experts who are not especially interested in the subject, but rather in the curriculum as a whole, and in its influence on the development of the child. It is believed that this proposal possesses merits which will make it appeal to critical and informed students of education, regardless of their subject-matter affiliations. If it cannot command support from such individuals, it certainly should not become part of the curriculum as a result of either more persistent vocalization or more skillful maneuvering on the part of its proponents.

Varieties of approach. Further disclaimer of the intention on the part of proponents of this material to expand a subject-matter field still further may be found in the fact that the suggested objectives can be achieved without resort to another course in the high school at all. It seems likely that the probability of securing the desired results would be greater if a new course for the high school could be set up. But a new course is not absolutely necessary. It is simply proposed that certain material, conveniently designated by the word "arithmetic," be taught on the senior high school level. There are at least four ways of accomplishing this. The first is through the medium of a course set up for that purpose. Such a procedure is most likely to be successful in a school with rather definite preferences for the organization of learning into the usual course classifications, and with the content of these courses well circumscribed by relatively rigid courses of study, custom, or teacher conservatism. A second method of achieving the desired results is to incorporate the material in other courses in mathematics. Such a procedure will necessitate a flexible organization of subject matter and a willingness to make careful comparisons of values. A third possibility is to include the material in other courses in which it might logically fit. It seems a bit difficult to understand just where some of the material later to be proposed will fit, but much of it can no doubt be worked into courses in economics, government, home economics, industrial arts, and others. In addition, many high schools are now experimenting with new courses which do not fit into the older categories very well, but which involve socially useful material which is deemed worth teaching. Many of these subjects have mathematical aspects which could be illuminated by a consideration of some of the topics indicated below. A fourth method of achieving the results intended, available only to those schools which happen to have a curriculum organization which ignores or greatly minimizes the importance of the older subject classification, is to present the material desired through the medium of activities which transcend the usual subject boundaries. Similar adjustments could be made in the use of a core curriculum or any of the many variations of the integrated curriculum gradually coming into experimental use in many progressive high schools.

The approach to the problem is theoretical rather than experimental. Very little experimentation has been found, the conclusions of which have implications for the problem at hand, but certain arguments seem to have significance and validity, and these will be examined.

At first thought the proposal advanced above seems to be a suggestion for revision of the grade placement of curriculum material. It is that and more. Since it is that in some measure, the broad problem of grade placement will be involved.

II. SPECIFIC SUGGESTIONS AS TO TYPES OF MATERIAL

Attention should be directed at once to samples of the kinds of material which are being suggested for inclusion in the curriculum of the senior high school. Practically none of the material herein suggested is the product of the writer's ingenuity. Most of it is collected from textbooks in arithmetic, textbooks intended for use in other branches of mathematics, books on the teaching of arithmetic or high school mathematics, and miscellaneous discussions of the teaching of arithmetic. Contributions to the development of the point of view underlying the presentation of some of this material have been made by many other writers, as, for example, by Buckingham and Buswell in the Tenth Yearbook of the National Council of Teachers of Mathematics, and by these and other individuals in many other writings, too numerous to mention. It should be noted, however, that many of these discussions have presupposed that the material suggested would be taught in the upper grades of the elementary school, or in the junior high school. No comprehensive survey of practice is available which would indicate the extent to which this is being done. The suggestion that this material should be taught in the senior high school need not deter teachers in the upper grades from making such contributions to the achievement of the objectives as conditions will permit. It should be noted that only "samples" of possible kinds of material are here presented. The classroom teacher reading this Yearbook would no doubt prefer a syllabus, or a comprehensive outline of proposed material. To write such a syllabus at this time would be impossible, and even to attempt it would be presumptuous. To expect such a specific suggestion is to misunderstand the purpose of the chapter. That purpose is to call attention to certain possibilities with reference to the inclusion of material commonly called "arithmetic" in the program of the senior high school and to illustrate the various types of material which are available.

Several types of material may be included in the arithmetic presented in the senior high school. Four of these will be discussed in turn. The first may be called for want of a better term, *applied* arithmetic. This may emphasize computation, or information, or both. The second type of material will be concerned with arithmetic as a science, as a branch of mathematics, as a system of ideas. Much of this material has values more cultural than practical; but it may well contribute to insight into many of the phenomena of life, as well as insight into the nature of arithmetic. The third type of material really represents a union of the ideas involved in the first two. Some study may be made of the everyday applications of number science—how an understanding of the nature of the number system and of number science meets certain needs of everyday life, and how the nature of the number system conditions the nature of arithmetical processes.

The fourth group of topics includes those which are presented primarily because of their interest values. They will occupy a more prominent place in an elective course than in a required course; they will bulk larger in those curriculums which are not primarily practical than in those which have a vocational emphasis. Many of these items should be found in optional, supplementary work of individual pupils.

Applied arithmetic. Consider first the possibilities in the field of informational and social arithmetic.

At this point some explanation may well be made relative to this term "applied arithmetic." Many readers will begin to think in terms of the "application of the four fundamental processes to concrete situations." But it is precisely this somewhat restricted conception of applied arithmetic which has been criticized as being somewhat too narrow to be maximally fruitful. It is erroneous to think that a person is "applying" arithmetic only when he is engaged in performing one of the four fundamental operations with a pencil, or, for that matter, performing it mentally. He is also applying his arithmetic when he selects the process to be used, when he makes rough approximations, or when he simply appreciates the mathematical relationships and mathematical implications in what he experiences or in what he reads.

This is all closely related to the matter of "meaning" in arithmetic, which is given considerable emphasis in this Yearbook. Further elaboration of this viewpoint is not necessary here, but will be found in other chapters. Suffice it to say that only that individual who has learned the meaning of number processes and number relationships is able to make the maximally effective use of "applied" arithmetic.

Consider first, as a sample, the subject of taxation. As pointed out in other chapters in this Yearbook, much of the material now taught in arithmetic is characterized by a type of approach much more characteristic of work in the social studies than in mathematics. The reader will note that much of the material here suggested fits this description. It is believed that the reader will also see possibilities in this material which can be realized somewhat better if the subject matter is studied in the senior high school than if an effort is made to interest children on the eighth grade level in it. Under the general topic of taxation such matters as the following may be studied:

1. Kinds of taxes.
2. Just what is taxed by each kind.
3. Who finally pays each kind of tax.

4. Passing taxes on to someone else, finally on to the elusive "ultimate consumer."
5. How taxes "pyramid."
6. Taxing various groups: consumers, manufacturers, the rich, the poor, motorists, middlemen, government employees, etc.
7. The political implications of various kinds of taxes. Why it is relatively easy to tax many people small amounts.
8. Tax-collecting machinery: state, federal, local.
9. Tax limitation laws and constitutional provisions: their value, their harm.
10. Taxes as fees for services.
11. Taxing frugality, thrift, enterprise, inventiveness.
12. How tax money is spent: a study of governmental budgets.
13. Taxes which save money; e.g., new fire fighting equipment may reduce insurance premiums.
14. "Hidden" taxes.

A similarly extensive study could be made of the subject of insurance, with attention given to phases of the topic which are now neglected for reasons indicated above. Attention could be given to such matters as the nature of an insurance estate, the disposition of the proceeds from an insurance policy, kinds of insurance suitable for special purposes, and the savings feature of life insurance policies.

As another illustration of a broad topic which might be of interest and value to high school pupils, we may consider the arithmetic of leisure. Much has been written on the subject of leisure in recent years, and it is generally believed that people have much more of it than ever before. There is much concern over the apparent disinclination of people to use this time in ways which certain persons consider important. Little attention has been given to the fact that to spend leisure time profitably and "constructively" costs money. Vacations are expensive approximately in proportion to their value in meeting acceptable criteria for the spending of leisure time. The following points are merely suggestive of a vast area which has been quite untouched in our curriculum:

1. Automobile touring vacations
    a. Planning auto trips
    b. Reading and using road maps
    c. Selecting routes
    d. Computing driving costs
    e. Computing living costs on the road

2. The cost of hobbies
    - *a.* Fishing: license, equipment, transportation, etc.
    - *b.* Stamp collecting: stamps, equipment, membership in stamp clubs, subscription to a journal, corresponding with others interested in the same thing
    - *c.* Attending baseball games: "following the team," admission, transportation, buying a program, buying newspapers to get the news
    - *d.* Playing golf: equipment, clothing, green fees, club dues, caddy fees, balls, locker fees, tips, transportation, wagers.

The whole field of consumer education has received so much attention recently that no discussion of the possibilities in this area is necessary in this paper. From the standpoint of arithmetic, the topic to be presented would probably be "The Arithmetic of Intelligent Purchasing." This will include such considerations as the purchasing of "dressed" *vs.* "undressed" chickens, forming accurate judgments relative to the size of containers of various shapes, the purchase of "ready-packed" *vs.* "counter-packed" ice cream, and a whole host of similar problems. The reader will be left to consider for himself other possibilities in this area. As a sample of an important topic, which some might consider within the area of consumer education, while others would not, we may suggest a topic, "The Arithmetic of Real Estate." Among the topics to be considered would be the following:

1. Factors affecting the value of land
2. Factors affecting the value of a building, especially a house
3. Rates of depreciation of houses
4. Ways of determining the value of property
5. The relative cost of owning and renting a home
6. Interest on mortgages
7. The legal status of mortgages
8. The resale value of property
9. Taxes and assessments on real estate
10. The arithmetic involved in house building: architect's fees, surveyor's fees, when to pay a contractor, where and how to borrow money, F.H.A. plan, cost of various plans of houses, planning the most economical heating system, etc.
11. How city property is surveyed, measured, and described.

Sufficient attention has not been given in discussions of the arithmetic curriculum to the cultural possibilities inherent in the study of the arithmetic of specialized vocations. There was much of that material in the older arithmetics, but most of it has been eliminated. It was originally included because of the prevalence of an economy in which many persons performed for themselves certain tasks which were expedited by a knowledge of the arithmetical topics in question. Papering offers an illustration. Formerly many persons hung their own wallpaper (some still do). In time the topic was eliminated from arithmetic textbooks because of the belief, perhaps correct, that not enough persons still hung their own paper to justify the inclusion of the topic. It was also believed that such of the pupils as did use this ability in later life would have forgotten what they learned in school by the time they needed it, or would be able to figure the thing out for themselves if they had not had the study. It was alleged that professional paperhangers did not make their computations according to the methods which the book used. The critics usually overlooked the irrelevance of this last point.

Interest here is in the informational values to be derived from the material in question rather than in the contribution which it may make to vocational effectiveness. It is not unthinkable that the items here under discussion may have value from a guidance point of view, and certainly some insight into how people in various occupations perform their duties is part of the equipment of an intelligent person. Only a few illustrations can be given. Some insight into the arithmetic which a farmer uses may well be part of the cultural equipment of the urban consumer of farm produce. The computation of butterfat yield by cows, the study of formulas having to do with mixing fertilizer ingredients, and the way various kinds of farm products are measured and weighed when sold, are suitable topics for investigation. At the moment of writing it is reported that the corn-hog ratio has changed so that it is more profitable for farmers to sell corn than to fatten hogs with it. An appreciation on the part of urban dwellers that one of the farmer's problems is to try to predict this ratio a year in advance might contribute to a more sympathetic attitude toward government efforts to alleviate the plight of agriculture.

Many occupations have their peculiar computational practices. Many of them offer opportunities to use arithmetical skills which we try to teach, and arithmetical concepts which we deem important. The motivational values here should not be overlooked. As a final illustration, printers like to use paper of such a shape that the ratio of the length to the width is equal to the square root of two. A sheet of this shape may be folded any number of times, and the resulting shape is the same. The values of an item of this kind in developing an appreciation of the mathematical relationship involved are at once apparent.

Another group of problems in the field of social arithmetic of a type beyond the comprehension of elementary school pupils may be found in the field of solid geometry. Most present-day textbooks in that subject contain many computational problems involving the measurement of solids. It is not necessary for the pupil to have memorized the relationships involved, or to be able to demonstrate their truth logically in order for him to make the computations and derive at least some of the appreciational value which the material contains.

Space does not permit more than a passing reference to the whole field of financial arithmetic. The mathematics of finance, of course, can easily go beyond the limits of a course in arithmetic. However, there are important sections of the subject which are primarily computational in their nature. Much can be done with the topic by pupils who have made no formal study of algebra; still more can be done by those who have studied the amount of algebra now normally expected as a part of the usual junior high school mathematics program.

Numbers and processes. Much is said in this Yearbook about teaching arithmetic for "meaning." There are many ways in which meaning can be brought into arithmetic teaching. It is especially important, both from the standpoint of broad cultural objectives and from the standpoint of facility in computation, that pupils be brought to a high level of understanding relative to the rationale of the computational processes. Teachers in the grades in which the manipulative skills are taught are now being urged to give the pupil as complete and comprehensive an insight into the nature of these processes as is possible. Many teachers find it very difficult to do this as well as they would like. It may be that insufficient mental maturity is responsible for this difficulty.

If a course in arithmetic is offered in the senior high school, some attention may well be given to this matter of the rationalization of the fundamental processes. First the teacher may attempt to determine to what extent the pupils possess a respectable comprehension of the reasons for carrying on these processes as they do. If investigation reveals that the pupils do not possess enough insight, some time and attention may well be given to the problem of remedying the situation. In any case it seems quite reasonable to expect a marked enrichment in the pupils' understanding of arithmetic as a result of giving some attention to this problem.

Much controversy can be stirred up by a suggestion that attention should be given to the general subject of number theory. It is not suggested here that the high school student should be given an introductory course in that difficult and involved branch of mathematics which is usually designated by that title. But some understanding of the nature of the number system and how it works, its internal structure and its functioning, may well be taught in schools. Some attention will be given to this in the elementary school if suggestions for presenting material which are made in other chapters of this Yearbook are followed. A study of the nature of arithmetical processes, and the numbers and number relationships which are utilized will lead to the objectives here proposed. Some study of a number system on a non-decimal base should be included. Ways of performing the ordinary computations different from those now usually taught often throw light on number relationships. Among these the following may be suggested: adding by writing sums for various columns in their entirety, thus avoiding carrying, then adding the partial sums; dividing fractions by reducing to a common denominator and then dividing the numerators; multiplying by reversing the order in which the digits in the multiplier are used; and performing long division by subtracting, as a calculating machine does it. These and many other variations of computational processes can throw light on the nature of computation and the number system.

Applications of number theory. Perhaps the nature of the number system can be better understood from a study of certain applied topics which have been included in the third classification. Certainly a study of logarithms and the slide rule would find a place among the topics proposed for study. Interpolation is a more common activity in everyday life than many realize; it is often necessary in reading a railway timetable and is a common mental activity among those who peruse road maps. This would indicate that its study should not be limited to the conventional uses which are found in the classroom when using mathematical tables. Attention has been called elsewhere to the understanding of the number system to be derived from a study of the use of letters on automobile license plates, and the various library classification systems.[1]

In this connection something should be said about the importance of providing some education in the applied mathematics of chance. We seem to be engulfed by a veritable wave of schemes of various kinds which depend for their effectiveness on the average person's ignorance of the most elementary notions of probability. Unnumbered individuals purchase "bank-night" tickets in order to participate in a drawing for a cash prize, who have slight interest in the picture being shown, or indeed do not attend the performance at all. "Pot-o'-Gold" schemes of various kinds are rampant. Slot machines take their toll of nickels and dimes from persons who can ill afford to lose the money. Radio programs cater to the public interest in getting something for nothing. It would seem that some actual experience in tossing coins, dealing cards, and studying permutations and combinations might contribute to social welfare by teaching children just how their interests are involved in these schemes. It is not to be hoped that gambling can be eliminated from the world by a revision of the school curriculum, but perhaps people can be educated to protect their own interests even when they gamble. A study of the mathematics of chance might also protect people from superstitions. It seems quite probable that one-seventh of all bad luck occurs on Friday, and that about one-thirtieth of it occurs on the thirteenth of the month.

Interest values. Under the fourth heading we may consider topics which do not classify otherwise, or which are of value chiefly for purposes of stimulating interest. Certain historical items having to do with arithmetic and the study of arithmetic may be mentioned; for example, Nicholas Pike's arithmetic included a section dealing with changing the currency of New Hampshire to that of Connecticut. "Foreign Exchange" apparently was closer home in those days than it is today. How the Romans computed with their unwieldly numbers might interest some youngster. A study of the abacus, Napier's rods, and other historical items might be included. Mathematical puzzles and mathematical recreations are available in unlimited numbers. Many pupils will be interested in noting everyday exceptions to the arithmetic taught in school. For example, on the financial pages 106.27 does not mean what it is usually considered to mean in textbooks. The 27 is the numerator of a fraction whose denominator is 32. Baseball standings are usually referred to as "percentages" although they are expressed to three places, and many newspapers do not bother to print the decimal point. Finally, is a "batting average" an average or is it a ratio?

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[1] Guy T. Buswell, "Deferred Arithmetic," *The Mathematics Teacher*, **31 : 195,** May, 1938.

Only a few suggestions have been made to indicate the wealth of material which is available to add interest to the study of arithmetic on a higher level than that of the elementary school. Skillful teachers will find many additional items. Other suggestions have been made for the presentation of a large amount of material which will have social value, and which it is believed cannot be adequately treated in the elementary school.

III. THE QUESTION OF GRADE PLACEMENT

Some may believe that arithmetic is an elementary school subject, and as such has no legitimate place in the high school. Such a position reveals a lack of knowledge of the history of the subject. A brief summary of this history may be appropriate at this point in our discussion.

With due regard for qualifications which are necessary for a complete understanding of the situation it may be pointed out that arithmetic was at one time a respected and honored subject for study in the college. In that famous classic of early American educational history, *New England's First Fruits,* we find arithmetic listed among the studies of the third year for Harvard College. Moving from the seventeenth century to the nineteenth, we find arithmetic listed as one of the required branches of study in the freshman year at Ohio University in 1825. Reverting for a moment to the situation in the seventeenth century, it seems quite reasonable that a requirement in arithmetic should have been included in the college curriculum. Not much is known about the secondary school programs of that day, but they contained very little besides the study of the classics. A complete program of the Boston Latin School for 1789 makes no mention of any branch of mathematics; neither is any knowledge of arithmetic assumed for entrance. Inglis says that arithmetic was introduced sometime between 1814 and 1828.[2] This statement should not be taken to represent the first appearance of arithmetic on the secondary level. But the conclusion seems justified that arithmetic was taught on the college level because little or none of it was learned before, and that little must have been of a very elementary type. Kandel reports that arithmetic made its first appearance as a college entrance requirement in 1745[3] and he refers to it as "common arithmetic." The academies introduced arithmetic as a subject of study somewhat earlier. Certain private tutorial schools seem to have made the study of this subject available as early as 1700. One description of a school listed among the studies available "Arethmatick, whole Numbers and Fractions, Vulgar and Decimal."[4] This emphasis is of course entirely understandable in light of the predominating philosophy of the academy.

No definite date can be given for the first appearance of arithmetic in the elementary school, nor for the time when it had come to be established as an elementary school subject. In an age of flexible curriculums, when adaptation to pupil needs was accepted practice rather than defensible theory, there was much variation in practice. Several brief quotations from Parker's book are instructive. In 1789 "arithmetic . . . was often taught in the elementary schools." "Very few teachers were competent to teach more than the fundamental operations." "The arithmetics of Nicholas Pike of Massachusetts were famous. Pike's complete work, containing five hundred twelve pages, with a rule for nearly every page, was used in the grammar schools *and universities.* An abridged edition issued in 1793 was intended for the elementary schools. . . ." "The curriculum of the American elementary school down to the American Revolution included reading and writing as the fundamental subjects, with perhaps a little arithmetic for the more favored schools."[5] It may be noted in passing that even in arithmetic an abridged edition of a text intended for college use was considered suitable for use in the elementary school.

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[2] Alexander Inglis, *Principles of Secondary Education*, p. 165. Houghton Mifflin, Boston. 1918.

[3] I. L. Kandel, *History of Secondary Education*, p. 122. Houghton Mifflin, Boston. 1918.

[4] *Ibid.*, p. 169.

Eventually, however, arithmetic came to be stabilized in the elementary school. It became one of the basic subjects of study, ranking with reading and spelling in importance in the minds of teachers and public. It practically disappeared from the high school. At least it ceased to appear in the form in which it had appeared previously, namely, as a first course, involving instruction in the fundamental operations together with their applications, for pupils who had received almost no previous instruction in the subject.

Division of labor. In the last century or two we have witnessed a curious division within the field of mathematics. Much has been written and much has been said about "correlated mathematics" or "fused mathematics." The division of mathematics into discrete branches, as it has been taught in American high schools for the past one hundred years, is a comparatively new development. Euclid's "Elements" included much material which we now classify as algebra and arithmetic. But somehow, high walls came to be erected between these branches of the subject, and the typical algebra course of 1900 was about as completely innocent of anything arithmetical as it could be. Many courses did not take up even the most obvious numerical applications of algebra, such as, for example, finding the product of 48 and 52 by the algebraic technique of finding the product of the sum and difference of two numbers. Geometry also was purged of all references to other branches of mathematics, and in many courses even numerical applications were ignored. Thus arithmetic came to be overlooked as suitable material for study in the high school, either as a separate subject or as a branch of the subjects of algebra or geometry, and so far as mathematics was concerned the high school devoted itself to the pursuit of these "advanced" subjects, leaving arithmetic to the elementary school. The elementary school, at the turn of the century, was strictly minding its own business, giving its pupils as extensive and comprehensive a mastery of arithmetic as the time available, the intellectual capacities of the students, and the limited knowledge of the dynamics of learning would permit, but carefully protecting them from exposure to those other mathematical studies, the teaching of which had come to be associated with secondary school practice. Thus there grew up the idea that arithmetic was properly an elementary school subject, and that it did not belong in the high school, while algebra and geometry were thought to be more suited to the mental powers of secondary school pupils. This somewhat unfortunate division of labor became accepted and continued to represent common practice until the inception of the junior high school movement made possible a new attack on the problem of the redistribution of subject matter. This, however, was more effective in bringing small amounts of algebra and somewhat larger amounts of geometry into the seventh and eighth grades, than in moving arithmetic into grades above these. Up to the present time the general mathematics movement has had relatively little influence on the grades above the eighth grade.

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[5] S. C. Parker, *The History of Modern Elementary Education*, pp. 83 ff. Ginn and Co., Boston. 1912.

IV. CONSIDERATIONS RELATING TO PROPOSED MATERIAL

Dichotomy. The title of this paper, and the general trend of the discussion up to this point, would seem to imply a point of view held by many high school pupils, namely, that *arithmetic* and *mathematics* are two separate and distinct, even though related, branches of knowledge. The proposal here made is that arithmetic be given more attention as a high school subject. It has been pointed out how before the beginning of the junior high school movement the division of labor between the elementary school and the secondary school with reference to work in mathematics was quite definite. Consequently, many pupils overlooked the fact that arithmetic was a branch of mathematics, closely related to these other branches. And an arithmetic-mathematics dichotomy arose which was not to be completely dissolved even by the correlation movement of the last two decades.

It should not be understood that the proposal in this chapter implies any such dichotomy. A suggestion is here made for the inclusion of certain specific subject matter. This material more closely resembles that which has long been taught in the elementary school under the title of arithmetic than anything which has generally been taught in the secondary school. It is believed that anyone after examining the material would classify it as arithmetic. But it does seem that the division above referred to has done much to prevent the acceptance of the material herein described as appropriate for the consideration of high school pupils.

As any reader will readily recognize, this dichotomy is more apparent than real. It developed as a matter of convenience. But if any consideration of algebra, or geometry, or other branches of mathematics can help in the understanding of the material proposed, certainly such consideration must be given. It happens that in only a few instances is the understanding facilitated by such consideration. Hence the material proposed will generally fall in the field of "arithmetic." And since it does it seems logical to propose the addition to the high school program of studies of a course in this subject.

Requirements. The question arises whether the study of the material herein proposed should be required of all pupils, or whether it should be offered for the benefit of those who elect to study it. That question must find its answer in terms of criteria which are usually applied in other similar situations. One of the difficult problems in the administration of the high school curriculum is that of relative values. This has two aspects, that dealing with the values of subject matter for individual pupils, whether strictly as individuals or as curriculum groups, and that dealing with the secondary school population as a whole. If the material here indicated possesses educational values superior to those possessed by other material which now occupies a place in the program, it should displace the less worthy material. However acceptable the generalization in the last sentence may be on its own merits, only a very naïve practitioner of educational administration would accept it as a solution of the problem. Vested interests, curricular inertia, tradition, interests of teachers, availability of instructional materials, theories of the nature of learning, ideas about the relation between education and vocational effectiveness, curious definitions of culture, prestige, college entrance requirements, pupil whims, and many other factors of varying degrees of relevance have their influence on the curriculum. However, a more functional point of view of the curriculum is gradually gaining acceptance by public, parents, pupils, and school administrators. Some may be so impressed with the importance of the material proposed in this chapter that they will wish to place it among the constants in the curriculum. Others will prefer to permit pupils to make their own judgments about the significance of the material for their educational programs. No suggestion will be forthcoming as to whether the work here suggested should be "required" or "elective." Most of it is not available at all at present. Perhaps the question of prescription can be left to the future. The material will need to be organized, instructional instruments devised, classroom methods invented and evaluated, and results carefully investigated, before it can be known whether a body of educationally valuable material sufficiently significant to justify its prescription has been proposed.

Grade placement. Can the arithmetic which is here proposed for introduction into the secondary school program be taught in the elementary school, where the subject already has a recognized place, and where the efficiency of its teaching is steadily being improved? This again brings up the whole matter of grade placement. In a sense the proposal to teach arithmetic in the general curriculum of the high school represents a radical departure from established practice. It is not the purpose here to discuss the problem of grade placement of topics in arithmetic; this question is thoroughly examined elsewhere in this Yearbook. The reader will do well to evaluate the proposals in this chapter in the light of what is said about grade placement in the discussion in Chapter III. However, since these proposals do in the last analysis constitute recommendations for the assignment of certain subject matter to new locations, responsibility for some discussion of the theoretical basis for such recommendations cannot be evaded at this point.

Careful examination of the concrete suggestions above will indicate that there are important aspects of arithmetic, the complete understanding of which depends to a large extent on the mental maturity of the learner. For example, it is doubtful that typical pupils in the eighth grade would be able to develop any genuine understanding of logarithms, or of some of the other topics suggested, although some superior pupils will. A certain amount of mental maturity seems to be necessary for the intellectual manipulations necessary for the understanding of various kinds of abstractions. Then there are other topics which seem to require social maturity, not only for understanding, but also for that amount of genuine interest which is necessary for good learning. An example may be found in the topic of insurance. No one would deny that children in the sixth or seventh grade might be able to understand some of the fundamental concepts involved in the study of this topic. However, there are other aspects which are sufficiently elusive to tax the powers of comprehension of the typical adult who is attempting to make that delicate balance of factors involved in the setting up of a practicable, and, at the same time, adequate, program of protection for his family. Since this topic is customarily given some attention in those courses in arithmetic which are offered in commercial curriculums in many high schools, it ought to be relatively easy to measure the effect of social maturity upon learning. This, however, has not been done, so we must again run the risk of taking a position based solely on good logic and on what seems reasonable. Consideration of other topics with important merits from the standpoint of social utility leads to a similar conclusion.

Needs. Is there a need for a course of the type under discussion? Before proceeding to a discussion of the ways in which such a need might be discovered if it exists, we should come to some agreement on what constitutes a need. Needs are relative. And the existence of needs is always a function of the point of view of the individual who is trying to recognize, identify, establish, or satisfy them, and of the educational philosophy against which they are projected. Hence, rather than make an attempt to establish a case in terms of needs, an effort will be made in this section simply to direct attention to certain situations which it is believed point the way to the desirability of the introduction into the high school program of some of the material which has been described.

All this does not mean, however, that such a need cannot be established with some degree of certainty, but rather that it is not convenient to do so at this time. Evidence may already exist which would establish the case; no systematic search for such evidence has been undertaken. It is likely that it does not exist in usable form. But evidence of the need for the introduction of this material into the curriculum will be derived from a comprehensive study of the failure of a large number of persons to make the maximally effective adjustment to their environment because of the absence of that material. Or it will be derived from a collection of instances in which individuals have found use in everyday life for those abilities and that information which the proposed material should provide.

One of the reasons for teaching arithmetic in the elementary school is that people may be able to perform computations which occur in the normal activities of the everyday lives of typical adults. Any reader can supply his own evidence, perhaps anecdotal in nature, to support the generalization that there are many adults in the world today who use their arithmetic with less than maximal efficiency. They may have learned the fundamental operations with integers, fractions, or decimals, at one time, but they show slight evidence of having done so. They stumble when they find it necessary to perform a simple computation under the observation of others, and frequently retire from the scene in embarrassment. Admitting for the moment that the elementary school is rapidly improving its procedures in the teaching of arithmetic, we must face the question, does the high school owe anything to the individual typified above? The arithmetic of college students has been investigated frequently, but only one sample of the results will be quoted. In one large state university "half of the students made scores below the eighth grade norms in operations with integers that involved problems in division. In other aspects of arithmetic the percentage below the eighth grade norms varied from 4 to 40. On one of the tests 14 per cent of the college students fell below the fourth grade norms!"[6] Many other instances of similar studies of the arithmetical equipment of high school pupils, college students, and adults, could be cited. In addition, there is need for further attention to the informational aspects of arithmetical topics. The lack of knowledge on the part of adults along these lines has not been so definitely established, but common observation indicates that it is as significant.

Cautions. Certain cautions may well be presented relative to the proposed material. Some of these apply with particular cogency in the case of a specific course set up for the purpose of achieving the results implied in this paper. In the first place, it is not proposed that the arithmetic of the elementary school simply be given a thorough review. There are some courses in arithmetic in high schools today which do not go beyond that. Sometimes the teacher has higher ambitions, but he seems to think that first things should come first, and that there is no use in attempting to build a superstructure on a shaky foundation. So he tests the pupils, finds them weak in the fundamental operations, and begins by giving a review. Frequently the semester is over before the review is completed. In the second place, the course must not wear itself out in a futile pursuit of "100% accuracy." Lest these proposals seem to contradict what was said above about the needs of adults for a better command of the fundamentals of arithmetic, let it be said that this contradiction is more apparent than real. Also, the course must not be one in number theory. Certain aspects of number theory are both interesting and useful in the education of the average adult. Suggestions were made for the inclusion of certain material along

[6] Alvin C. Eurich, in *General Education in the American College, Thirty-eighth Yearbook of the National Society for the Study of Education*, Part II, p. 81. Public School Publishing Co., Bloomington, Ill. 1939.

this line. However, the mathematically trained teacher may be tempted to approach this material from such a highly theoretical point of view as to defeat the purposes intended. This is not so likely to happen if the material is not presented in a special course, but in connection with other courses. In the fourth place, there is some danger that the proposed course may degenerate into a course in "rapid calculation." Such courses used to be taught, especially in commercial schools in the days before bookkeepers had the benefit of adding machines. Drill materials for this type of work are still available, and no doubt are used in some courses in "Commercial Arithmetic." This point of view may be appropriate in its place, but does not belong in the treatment of the material assumed in this paper. Finally, the course must not become a series of complicated and difficult verbal problems, dealing with socially unimportant situations, but interesting chiefly because of the demands which they make on the learner for ingenuity and dogged determination in finding a solution. The point of view here referred to was sometimes present in the one-room rural school of the nineteenth century, when the teacher was possessed of a high intellect and a liking for mathematics, together with a consummate faith in the disciplinary value of the involved arithmetical problems which he assigned such of his advanced pupils as were interested in attempting to do the work. Such a teacher frequently possessed a private library of pet problems with which he baffled his students and impressed the public, and often embarrassed other teachers who did not possess his interest in that particular form of intellectual activity. Many such problems found their way into early textbooks. An example of such a problem, taken from a book published in 1880, follows:[7]

> If 15 men cut 480 sters of wood in 10 days, of 8 hours each, how many boys will it take to cut 1152 sters of wood, only 2/5 as hard, in 16 days, of 6 hours each, provided that while working a boy can do only 3/4 as much as a man, and that 1/3 of the boys are idle at a time throughout the work? Ans., 24 boys.

[7] Joseph Ray, *Ray's New Higher Arithmetic*, p. 187. Van Antwerp, Bragg, and Co., New York, 1880.

Chapter VII

THE SOCIAL PHASE OF ARITHMETIC INSTRUCTION

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In the modern school the primary purpose of the curriculum should be the provision of a series of learning experiences that will develop in the learner constantly enriched social insight and understanding. To evaluate a particular learning experience we may apply such criteria as the following: In what ways does this experience make more meaningful to the learner the present social situation and help him to interpret it? How does the unit lead the individual to see the contrasts between the present status of practices in an important area of human endeavor and conditions in that area in the past? How does it lead the learner to see the part that human intelligence has played in the solution of problems and difficulties that have arisen in the evolution of social institutions? How does the experience bring to the attention of the learner the emerging problems and difficulties in an important area of human affairs and the ways in which it is proposed to solve these questions? Does the experience lead in an increasing degree to the development in the individual of an appreciation of the need of his active participation in the redirection of social affairs through the exercise of intelligent human control?

It is the purpose of the present chapter to consider the implication of these criteria for arithmetical instruction. They reveal the necessity of considering the possible contributions arithmetic can make to the social function of the school. At the same time it will be pointed out that these considerations do not in the least obviate the need of providing fully and adequately for effective training along definitely mathematical lines.

THE RELATION OF THE MATHEMATICAL AND SOCIAL PHASES OF ARITHMETIC

The importance of the above named criteria in connection with the arithmetic curriculum has been clearly recognized in the preliminary report of the National Council's Committee on Arithmetic.[1] In this report the Committee took the stand that "the functions of instruction in arithmetic are to teach the nature and use of the number system in the affairs of daily life and to help the learner to utilize quantitative procedures effectively in the achievement of his purposes and those of the social order of which he is a part." This point of view recognizes two major mutually related and interdependent phases of instruction in arithmetic, namely, the *mathematical* phase and the *social* phase. Full recognition of both phases is essential. Emphasis on the social phase to the neglect of the mathematical phase will not develop in the pupils the quantitative concepts, understandings, and insights that should be the outcomes of a well-rounded program of instruction in arithmetic. On the other hand, emphasis on the mathematical phase to the neglect of the social phase will not lead the learner to sense completely the social significance of number in the institutions and affairs of daily life. A balanced, well-integrated treatment of both phases is essential. Arithmetic should be both mathematically meaningful and socially significant. This is essential if arithmetic is to make its maximum contribution to the development of socially competent individuals.

The evolution of the arithmetic curriculum. Arithmetic was introduced into the schools of this country because of its "practical values in business." The first textbooks contained business applications which represented definite needs of the period, almost always on the adult level. With the passing of time new applications were introduced, but the school, always a conservative institution, retained almost everything that had in the past been taught. Gradually the arithmetic curriculum developed into an unwieldy mass, containing a body of content, much of which was useless and impractical, and with little meaning to the pupils because they never encountered in their daily experiences many of the processes and topics taught. Arithmetic was usually taught by people who had had very little, if any, business experience. Academic-minded persons began to emphasize the historical position that arithmetic is "the science of numbers," and to stress the purely mathematical phase of the subject to the neglect of the social phase. Gradually the possibilities of the science of numbers were developed by those who were in charge of our classrooms and by those who wrote our textbooks. Thus, the purely mathematical phase of the subject became the basis of the major content of the arithmetic curriculum. It was inevitable that arithmetic, which was introduced into our schools because of its practical values, should therefore become more and more academic and unrelated to the affairs of daily life. Its continuance in the curriculum was increasingly justified on the grounds of mental discipline rather than its practical utility. When the theory of mental discipline was exploded, other values such as "cultural" values, "preparatory" values, "conventional" values, and "leisure-time" values were substituted. To a considerable degree these values still dominate the work in arithmetic in many of our schools.

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[1] R. L. Morton, "The National Council Committee on Arithmetic," *The Mathematics Teacher,* 31 : 267-272, October, 1938.

The changing arithmetic curriculum. The present century has been marked by a number of attacks on the arithmetic curriculum which have greatly modified it. In the period from 1910 to 1920 numerous studies were made by various investigators, among them Wilson, Jessup, Bobbitt, Woody, and Charters, to determine the arithmetic that was useful in the affairs of the daily lives of adults. These studies revealed the fact that actual social usage of number processes included a much narrower and simpler range of skills than those commonly found in current courses of study. Many topics dealing with the applications of number were found to be obsolete and useless. Unfortunately, these studies dealt with adult needs rather than child needs. As a result of this group of investigations, the arithmetic curriculum has been greatly simplified, and much of the accumulated dead wood has been eliminated. Topics such as cube root, surveyor's measure, complex fractions, and many others of no social utility today except for workers in specialized occupations are no longer found in up-to-date courses of study and textbooks. The place of these obsolete topics is being taken by rich units of social experience in which the pupils may learn about important social institutions, and, at the same time, practice increasingly mature and refined mathematical procedures and technics for dealing more effectively with the quantitative aspects of their daily affairs.

Another group of studies dealing with various aspects of child development has also led to significant changes in the arithmetic curriculum. It was early demonstrated that above grade two more pupils failed in the subject of arithmetic than in any other area of the curriculum. Studies of factors thought to be contributing to this situation led to the conclusions that (1) instructional materials were not effectively organized, (2) many of the processes were being taught at levels in the school at which many of the pupils did not have the mental ability needed to master them, and (3) instruction tended to emphasize unduly the computational phase of arithmetic with the result that much of the work in the classroom was without significance to the pupils. The need of more adequate provisions for individual differences in readiness for the learning of various topics, in rates of learning, and in difficulties encountered was also revealed. As a consequence of these investigations much has been done to improve the quality of instructional materials, to adjust the gradation of topics more nearly to the facts known about child development, and to enrich and socialize instruction so that the work in arithmetic may be made increasingly more significant to the pupils.

The child psychology movement has also led to the recognition of the need of giving more adequate attention to what the National Council Committee has called the "social phase" of the arithmetic curriculum. Undoubtedly one of the chief problems of the school is to determine from the wide range of applications of numbers those that the great majority of people can profitably use in their daily affairs. An important section of these social needs and problems about which our youth should be made intelligent consists of those that appear in the normal, desirable activities of boys and girls both in and out of school, an area as yet practically unexplored. The writer[2] has shown that analyses of current social problems as seen by frontier thinkers in the fields of economics, sociology, and political science contain many problems that are closely identified with topics that are now included in the arithmetic course: for example, issues related to taxation, to the consumption of goods, to the distribution of wealth, to wise expenditure of funds, and to the support of the indigent, handicapped, and unemployed. The selection of suitable and significant topics dealing with social processes in which number functions directly as a basis of units of instruction at various levels of the school is strongly recommended by this Committee.

At the same time consideration must be given to the organization and gradation of the basic curriculum material so that the pupils will have the mental ability required to master the units of work at the time they are introduced, and so that the work will be within the range of their experiences and interests. The recognition of the fact of maturation of ideas, skills, and concepts requires that the instructional programs be so organized as to contribute to the continued development of understanding of, and insight into, mathematical relationships as the pupil advances from stage to stage through the school.

Attainment of the mathematical aim of instruction in arithmetic is regarded as possible only if meaning, the fact that the children see sense in what they learn, is made the central consideration in arithmetic instruction at all stages. As is indicated in the first report of the National Committee, "Arithmetic is conceived as a closely knit system of understandable ideas, principles, and processes, and an important test of arithmetical learning is an intelligent grasp of the number relations together with the ability to deal with arithmetical situations with proper comprehension of their mathematical significance." The desired outcomes, related to the mathematical and the social phases, can most likely be secured by making certain that the pupil has vital social experiences in which he is led to see the number involved in situations that are meaningful and significant to him, and if at the same time the teacher takes the steps needed to make certain that the mathematical meaning[3] of the number elements involved is fully grasped by the learner. The teacher should plan definitely and systematically to develop number meaning by making the pupil conscious of the ways in which number functions in the experience and then seeing to it that insight into the quantitative procedures is developed. The aim of the teacher at all levels should be to develop in the pupils the ability and the disposition to view the affairs of life in orderly, systematic ways, and to use quantitative technics, when feasible, to enable them to see the relationships involved more clearly and exactly. This mode of quantitative thinking can be refined and extended as the learners progress through the school.

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[2] Leo J. Brueckner, "Social Problems as a Basis for a Vitalized Curriculum in Arithmetic," *Journal of Experimental Education,* 1 : 320-322, June, 1933.

THE SIGNIFICANCE OF THE SOCIAL PHASE OF ARITHMETIC

John Dewey has defined education as "that reconstruction or reorganization of experience which adds to the *meaning* of experience and which increases ability to *direct* the course of subsequent experience." In this statement there are two important words, namely, "meaning" and "direct." As far as arithmetic is concerned, the word "meaning" has a two-fold connotation. It refers on the one hand to the meaning of number itself, an understanding of the number system and its interrelationships. On the other hand, "meaning" refers to the social significance which the situations in which number is applied have for the learner. For example, the institution, taxation, involves principles, issues, and practices which do not require the direct manipulation of number processes to grasp their social significance. A class might debate the issue, "Should Minnesota adopt the sales tax?" without going at all into the actual computation of the taxes, but by weighing carefully the various social considerations involved. On the other hand, a class might devote the whole time allotted to taxation to the actual computation of taxes and tax rates and devote no time whatever to the consideration of the social aspects of the topic. If the pupils were not led to consider the social significance of this topic in the broadest sense of the term "meaning," as here presented, it is obvious that they would not become aware of the issues related to the problems or be made familiar with ways in which they are being dealt with today. As a consequence they would not be prepared in any way to "direct" more effectively the course of their subsequent civic experience involving the consideration of issues related to taxation and would not be able to participate intelligently in the solution of the issues involved. If this topic is dealt with by the teacher in such a way as to bring out the implications of the questions listed as criteria in the first paragraph of this chapter, it is obvious that not only will the topic have significance for the learner but that he will also later be able to participate intelligently in the consideration of the issues involved and in steps leading to their solution. This social approach must, of course, be paralleled by well-planned steps to bring out clearly the mathematical elements and relationships that are involved.

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[3] Arithmetic instruction which features the mathematical phase as a means of developing meaning is discussed in other chapters, particularly those by Sauble, Thiele, and Wheat. These chapters supplement Dr. Brueckner's chapter, which deals specifically with the social phase. (*Editorial Board*)

Major points of emphasis in the arithmetic instruction. The topic, taxation, or any similar topic, would, of course, be incompletely treated unless the pupils were led to consider it adequately from both the mathematical and social points of view. The treatment of such topics, in so far as the mathematical phase is concerned, would include such items as the making of essential computations, the bringing out of the relationships between the quantitative factors concerned, the presentation of the basic symbolism and units of measure involved, the interpretation of necessary or pertinent tabular or graphic data, and a consideration of the validity and limits of existing knowledge about the topic under consideration. In the lower grades the comprehensive study of such a topic, as "What items affect the cost of mailing a letter?" offers ample opportunity to bring out both the ways in which number helps us to manage human affairs and the varied mathematical relationships in a simple social situation. The general information gained by the pupils in the consideration of such a topic is also of undoubted value in helping them to understand some of the aspects of the social process, communication. It should be clear that in some units of work there may be somewhat more emphasis on the social than on the mathematical phase of arithmetic, while in other units the reverse will be true.

It may be helpful if at this point some of the major ideas that are related more directly to the social phase of arithmetic are more definitely identified. These ideas, sometimes called themes, may all be brought out in some way in any well-selected unit, although this is not essential, since the idea or ideas that may emerge in any unit will be determined to a large extent by the scope of the topic and by the way in which it is explored by the class under the guidance of the teacher.

Social Evolution. To add to richness of meaning there is no good reason why the pupils should not be led to see that both our present social institutions and also our ways of dealing with their quantitative aspects are the more or less perfected end products of a process that is in general evolutionary and progressive. This applies not only to the development of our number system itself, but also to the study of such human institutions as methods of measurement, money and barter, taxation, and insurance. The pupils should be led to see that these and many other institutions began with crude methods and procedures that have since been improved, refined, and standardized through the application of human intelligence to the task. The study of this evolutionary process will lead the pupil to see the significance of number in human progress and should lead him to see that this developmental process is not completed at the present time.

Such topics as the following are rich in their possibilities for developing the idea of social evolution:

1. Ways of telling time, past and present.
2. The kinds of money that people have used and use today.
3. The development of our units of measure.
4. How our present system of taxation developed.

Cooperation. The pupils should be led to see how effectively number has contributed to the development of inter-cooperation among people. As Dr. Judd has pointed out, if we merely teach the pupils how to tell time by the clock and neglect to point out at the same time the ways in which the clock has facilitated inter-cooperation among peoples, we have overlooked a very valuable contribution that a consideration of this sociological phase of the topic might make to an understanding of the social process. In the same way the treatment of the topic—money—should not be limited to mere computation with money. The topic can be presented in such a way as to show to the pupils its value as a device invented by man as a more manageable way of exchanging and distributing goods and paying for services rendered than the earlier social process of barter and exchange. Likewise, the teaching of taxation, insurance, banking, our credit system, and many other topics, should not fail to stress the fact that each of these institutions is an illustration of the ways in which number has facilitated inter-cooperation among people.

The following topics suggest units rich in possibilities of developing the idea of inter-cooperation:

1. How the clock helps us to live together.
2. What is the system of numbering houses used in our community?
3. Where does the money for our schools come from?
4. How are people paid for the work they do?

Invention. The study of the process by which human intelligence has in the past devised improved procedures for dealing more effectively with quantitative aspects of social affairs should have as its goal an awareness by the pupils that new problems will as surely arise in the future as they have arisen in the past. The pupils should be led to recognize the need of inventing new and improved ways of dealing with those aspects of their lives that can only be managed efficiently through the use of quantitative technics. Ingenuity in dealing with quantitative aspects of situations should be fostered. "The progress of science depends very largely upon the facility with which facts can be recorded and relationships between them considered. The notation of the mathematician affords the maximum of precision, simplicity, and conciseness."

The following topics a[illegible]uggestive:

1. What is the metric system and why was it devised?
2. How do we measure food value?
3. What new forms of taxation are being considered today?
4. Why have consumers' cooperatives been established?

Control over Nature and Natural Processes. Man is constantly engaged in a struggle to control nature and to direct natural processes to his advantage. Quantitative procedures have been of great assistance to him in this connection. The mariner and the aviator can proceed more safely because of the compass. The thermometer enables man to make necessary adjustments to fluctuations in temperature and to some extent to control it. The precise rigorous tests in experiments which determine courses of action are possible only because of the use of number. The weather bureau uses quantitative methods to predict the climatic conditions. Index numbers of production, distribution, and consumption are increasingly being used to aid in setting up measures to regulate natural processes. Insurance is possible because of the fact that we can predict mortality of individuals on the basis of carefully collected data. These and many other similar illustrations of human efforts to utilize quantitative procedures to direct natural processes to social ends show ways in which the teacher of arithmetic can develop mathematical meaning and ensure at the same time an awareness of the social significance on the part of the pupils of the items being studied.

The following topics are suggestive:

1. In what ways has the thermometer helped the man in the greenhouse?
2. How can we measure the difference in plant growth due to the effects of fertilizers?
3. What is being done by our government to control the production of crops?
4. How does the weather bureau help aviators and mariners?

Economic Literacy. Arithmetic affords a convenient vehicle for teaching the pupils significant facts about the production, distribution, and consumption of goods and about many of the other economic aspects of life, including insurance, business relations, banking, taxation, and so on. The pupils should be led to see the necessity of accurate, dependable information in the buying and selling of food, clothing, and other necessities of life, and in the management of their daily affairs, including, later on, their occupations. They should be taught about elementary business practices and the economic interdependence of human institutions. Sources of loss, such as fraud, forgery, speculation, damage to property, and bad debts, offer a fertile field for worthwhile discussions. As Dr. Horn has stated, the pupils should be led to "think while reading" and to evaluate the authenticity and dependability of information presented to them in any verbal or visual form. It is an unfortunate fact that in almost none of our present courses for teachers of arithmetic is adequate consideration being given to the need of educating individuals who have genuine economic literacy.

The following topics are suggestive:

1. How is the money we pay for stamps used?
2. Which is the cheaper way in which to buy, in bulk or by the package? Why?
3. Why do prices of commodities usually fluctuate from place to place and from year to year?
4. Is the money raised by taxation being spent economically and to the greatest advantage?

Kinds of units in arithmetic experience. The experiences that can be utilized by the teacher to achieve the purpose of arithmetic instruction are of various sorts. The wise teacher will seize every opportunity that may arise in an incidental way either in the classroom or in life outside the school to bring to the attention of pupils the social significance and the utility of number and its applications in the affairs of life. However, because of the unpredictable nature of these experiences and because of their unsystematic, disorganized occurrence, the teacher should also actively seek and plan other social situations which will be rich in the use and social application of quantitative procedures, and which at the same time are most likely to contribute to the development of the major ideas described above that are related to the social phase of arithmetic. In selecting these more systematic units of instruction the teacher should consider carefully the stage of the development of the arithmetic ability of the pupils, their mental level, and their needs and interests. It is not essential that these social situations require the use of the computational processes that are being learned by the pupils. However, it is obvious that the most desirable unit of instruction is a rich unit of social experience likely to appeal to the pupils as worth while that gives them contacts not only with applications of numbers in the affairs of life but also with the elements of computational arithmetic that are most appropriately taught to children at their stage of development. Units selected on this basis not only show the pupils the need for the drill work to be done to ensure mastery of the basic skills, but also vitalize the instruction by revealing the social significance of the processes or topics that are being learned. Insight into the social situation should grow out of a careful consideration of the relationships between the various elements in the experience which can be brought out sharply and definitely by the application of quantitative procedures. At the same time there should develop insight into the mathematical procedures that are being employed.

Sources of units stressing the social phase of arithmetic. Modern textbooks are giving more adequate consideration than in the past to the social phase of the arithmetic curriculum. For example, a recent study of Gustafson[4] of the amount of space devoted to strictly informational material related to the social phase of arithmetic showed that in two third grade books published in 1893 and 1895 there were only 69 and 74 lines respectively of this kind of material. On the other hand, in three third grade books published in 1935 the numbers of lines of informational material were 384, 523, and 1,089 respectively. These results are typical for a total of 22 third grade books. The newer books evidently contain much more informational material than the old books, although large differences exist from book to book. It should be emphasized that teachers should always feel free to go beyond the limits of the book for illustrations and applications of number. Well selected experiences that utilize direct contact with local situations are undoubtedly of greater value than those that are presented formally in a textbook.

In another study of similar informational social material, in this case involving no computation, Herrmann[5] found that in the books for grades four, five, and six, of five modern series,

[4] Anne Gustafson, "A Comparison of Activities Presented in Third Grade Arithmetic Textbooks for Developing the Informational and Sociological Functions of Arithmetic." Unpublished records on file at the University of Minnesota, Minneapolis.

[5] J. M. Herrmann, "A Study of Activities in More Recent Arithmetic Textbooks, Intermediate Grades, for Developing Informational and Sociological Functions of the Subject." Unpublished Master's thesis, University of Minnesota, Minneapolis.

the total amount of line space for the books of each series varied from 1,227 lines to 3,536 lines, a ratio of almost one to three. It seems evident that the amount of assistance the teacher can secure about social applications of arithmetic varies widely with the textbook.

Herrmann's analysis of subject matter revealed a wide variety of different topics that deal with the social phases of arithmetic in the books he analyzed. For purposes of illustration the fifty-four topics related to consumer education that he listed will indicate the nature and social significance of topics in this field that are now finding their way into textbooks. The list follows:

Buying and Selling Practices

1. Methods used in price labelling
2. Finding costs in a project
3. Widths in which cloth is sold
4. False bottoms in measures
5. The sales slip
6. Reasons for price changes
7. How to compute real profit
8. Basis of credit
9. Functions of better business bureaus
10. Early period prices and why they changed
11. Per capita food costs of countries compared

Using Transportation Facilities

1. Parcel post and express services
2. Parcel post insurance
3. Meaning of C.O.D.
4. Commuter's and round-trip tickets
5. How freight rates are computed

Banking

1. Making deposits and use of bank books
2. Savings account routine
3. Services of banks
4. Thrift by small savings
5. Check writing, endorsing, and cashing
6. Check stub records

Spending the Family Dollar

1. Patronizing the cafeteria
2. Choices in purchases for thrift
3. Cost of cut flowers and potted plants
4. Bulk buying advantages

5. Magazine subscription savings
6. Library fines
7. Rent
8. Homemade and commercially made costs compared
9. Grades and price of milk and gasoline
10. Quantity buying savings
11. Saving at sales
12. Advertisements
13. Checking sales slips
14. Accounts of receipts and expenditures
15. Time and cash payment plans
16. Car operation costs
17. Family budgets
18. Monthly bills and forms
19. Receipt forms and uses
20. Cold storage food costs
21. Food container costs
22. Electrical device operation costs
23. Heating costs compared
24. Sizes of cans of preserved foods
25. Large and small package costs compared
26. Water, gas, and electric rates
27. Cash discounts
28. A la carte menus
29. Secondhand buying
30. Charge accounts
31. Out-of-season food costs

Miscellaneous

1. Safety practices
2. Testing seeds for germination
3. Cost of window glass

An analysis by the reader of the topics listed above should make it clear that there is a definite tendency today to include in arithmetic instruction the kinds of topics that are likely to develop in the pupils an understanding and an appreciation of vital social practices and procedures and of the ways in which number functions in the affairs of daily life. Questions and problems based on such topics as these are certainly more valuable than the isolated problems that in the not distant past were the only kinds included in our textbooks.

Experimental development of curriculum units. Educational literature contains a number of descriptions of units that were taught in such a way that they dealt with both the mathematical and social phases of arithmetic. For example, Schaeffer[6] described a fourth grade informational unit on time measurement that divided the subject into four main topics as follows:

I. How the cave men told time
  1. The shadow on the rock
  2. The shadow clock
  3. The rope clock
  4. The flower clock

II. Clocks from long ago to now
  1. The sun dial
  2. The water thief
  3. The time candle
  4. The sand glass
  5. Early mechanical clocks
  6. Clocks with pendulums
  7. Smaller clocks and watches
  8. Electricity and clocks

III. How the world gets its time
  1. The causes of day and night
  2. The lines on the globe
  3. Setting the clocks
  4. Standard time
  5. Daylight standard time

IV. The story of the calendar
  1. What the moon told men
  2. Calendar from long ago to now
  3. Calendar reform

Schaeffer prepared a number of authoritative mimeographed articles covering all the topics which formed the basis of much of the class discussion. Twenty class periods were used to develop the unit. The following list of activities shows the nature of some of the lessons: "Simple discussion of what pupils already knew about the topic; work-type reading lessons, in which the pupils read articles to answer questions, to prove statements, to outline points, etc.; preparation of reports; giving of reports; planning of activities; making of replicas of early time pieces (shadow clock, sundials, water clocks, etc.); coloring maps to show time zones; solving clock problems; and reading timetables."

Tests involving both mathematical and social phases of the subject were administered at the end of each of the four major topics. The reader can easily see from an analysis of the contents of the unit how the major social ideas, evolution, cooperation, invention, and control over nature and natural processes, as well as the mathematical principles involving precision, computation, functional relationships, and symbolism, were all developed in an interrelated, meaningful way.

Records were also kept of the reactions of the pupils during the course of the unit. Schaeffer describes the outcomes of this unit as follows: "(1) The subject of time measurement was of genuine interest to this group of pupils during the entire unit. (2) The reading materials of the unit were not too difficult. (3) Illustrative materials, such as odd clocks and pictures of early time pieces and calendars, are available. (4) A large percentage of the pupils were able to understand such terms as 'standard time,' 'calendar reform,' 'daylight-saving,' 'pendulum,' 'rotate,' 'time-recording,' and 'telescope.' (5) Activities of special interest were the making of early time pieces, the giving of reports on materials read, the reading of stories related to the unit, dramatization of stories, and related art work." Obviously there were many opportunities to utilize quantitative procedures in this unit of work. However, the major emphasis was on outcomes more closely related to the social than to the mathematical phase of the subject.

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[6] Grace Schaeffer, "An Informational Unit in Time," *Elementary School Journal*, 38: 114-117, October, 1937.

Harap and Mapes[7] conducted a group of studies whose primary purpose it was to determine the extent to which children can master arithmetic processes through their use in lifelike activities and social situations so selected that opportunity for the use of number processes was certain to arise in them. One interesting group of units that was selected so as to bring in applications of decimals was a series dealing with the making of household preparations, such as tooth powder, furniture polish, ink, hand lotion, and paste. The only actual computations performed were those needed to carry on the activity. All work of the pupils was kept in notebooks which were frequently checked by the teacher. All errors had to be corrected. There was no drill as such. These units were taught in such a way as to contribute to the education of consumers, an important aspect of the social phase of arithmetic. Relative costs of homemade and commercially made products were compared. This led to interesting discussions of some of the social issues that arose, such as reasons for differences in costs. Harap and Mapes report that the processes involving operations with decimals were mastered by the pupils. Unfortunately no detailed data were given in the reports as to the range of socially significant information other than computational skills that the pupils actually acquired as a result of this work. The implications of the report of the study are that these informational outcomes were rich and significant. Similar studies involving the use of social experiences were also conducted by Harap and his associates[8] in grade three and in grade five. In each case it was amply demonstrated that not only were the computational processes ordinarily taught in these grades mastered at least as well as in ordinary classes, but that there were also rich concomitant values in the field of social understanding and insight.

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[7] H. L. Harap and E. Mapes, "The Learning of Decimals in an Activity Program," *Journal of Educational Research*, 29 : 686-693, May, 1936.

These studies and others that could be listed are indicative of the valuable contributions that can be made by a well-organized program of instruction in which full recognition is given to the need of considering both the mathematical and social phases of arithmetic. There is a great need of further exploration of the possibilities of rich social units of this kind. The teacher in the classroom who has a grasp of the significance of the point of view here presented can make real contributions to the improvement of the arithmetic curriculum.

In his book *Mankind in the Making*, H. G. Wells made the following statement which summarizes very well the point of view that the chapter has discussed: "The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity, and a means of expression, more exact, compact, and ready, than ordinary language. The great body of physical science, a great deal of the essential facts of financial science, and endless social and political problems, are only accessible and only thinkable to those who have had a sound training in mathematical analysis. The time may not be very remote when it will be understood that for complete initiation as an efficient citizen of the new great complex world-wide states that are now developing, it is necessary to be able to compute, to think in averages, and in maxima and minima, as it is now understood to be necessary to be able to read and to write."

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[8] H. L. Harap and E. Mapes, "The Learning of Fundamentals in an Activity Program," *Elementary School Journal*, 34 : 515-525, March, 1934.

Chapter VIII

ENRICHMENT OF THE ARITHMETIC COURSE

Utilizing Supplementary Materials and Devices

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This chapter aims to point out the potentialities that exist for vitalizing and enriching the teaching of arithmetic on all grade levels through the use of a wealth of supplementary instructional materials and devices which may be provided through the initiative of the individual teacher. Suggestions are given to assist the teacher in obtaining and using such materials.

The different types of supplementary materials and devices which will be illustrated and discussed in conjunction with the teaching of definite mathematical and social concepts include the following:

(1) Concrete materials which may consist of real objects or representations of real objects which pupils may manipulate in gaining first-hand experiences.
(2) Measuring instruments which may be provided for pupils or which may be constructed by pupils.
(3) Pictures—provided for pupils or made by pupils to represent mathematical ideas.
(4) Charts, diagrams, and graphs.
(5) Business forms used in the community—checks, deposit slips, withdrawal slips, monthly statements, bills, insurance policies.
(6) Advertisements, handbills, clippings, catalogs, pamphlets, which contain information which may be useful in illustrating or formulating current arithmetic problems.
(7) Posters, displays, exhibits, scrapbooks.
(8) Trips, interviews, special reports.
(9) Dramatizations of business procedures.
(10) Analyses of life situations having quantitative aspects.

Supplementary materials as well as other types of materials contribute to the two aspects of number which Buckingham designates by the terms *significance* and *meaning.* He states:[1] "By the *significance* of number I mean its value, its importance, its necessity in the modern social order. I mean the role it has played in science, the instrument it has proved to be in ordering the life and environment of man. The idea of significance is therefore functional.

"On the other hand, the *meaning* of number, as I understand it, is mathematical. In pursuit of it we conceive of a closely knit, quantitative system. . . . Under the heading of meaning I include, of course, the rationale of our number system. The teacher who emphasizes the social aspects of arithmetic may say that she is giving meaning to numbers. I prefer to say she is giving them significance. I hasten to say, however, that each idea supports the other."

Although it is impossible to discuss one aspect of arithmetic teaching without some reference to the other aspect, an attempt will be made to focus attention upon materials and devices which may be employed in the development of meanings of numbers, processes, and measures in part I of this chapter and reserve part II for special emphasis upon the supplementary materials and devices which may be used in giving social significance to numbers, processes, and measures.

I. DEVELOPMENT OF MATHEMATICAL MEANINGS

Developing meanings of whole numbers. Counting objects provides one of the first activities in the development of number meanings. In school and out of school, pupils count real objects for definite social purposes. They count the children on each side in a game, count the children present each day, count the scissors, books, chairs, and pencils, to see whether these are enough for the class. However, to develop adequate number ideas, pupils need many experiences with groups of objects beyond the activity of counting them in some given arrangement. As the next step, each pupil should be given the opportunity of experimenting with groups of objects in making as many different arrangements as possible. A pupil who has eight cardboard pennies may be encouraged to work out for himself and report to the class many arrangements which he will gradually come to generalize as the basic combinations. Eight will mean two 4's and four 2's as well as 6 and 2, 5 and 3, 7 and 1.

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[1] B. R. Buckingham, "Significance, Meaning, Insight—These Three," *The Mathematics Teacher,* 31 : 26, January, 1938.

Pupils need to use groups of objects in making comparisons also. Eight pennies are how many more than 6 pennies? Four chairs are how many less than 7 chairs, etc.

In making provision for this variety of experiences with groups, two types of concrete objects are required—objects small enough for pupils to use individually at their own desks, and objects large enough to be used at the front of the room in carrying through some activity which all pupils are to observe.

Small objects which may easily be provided in a large enough quantity so that each pupil may have a supply to work with include the following:

Play pennies—which may be purchased commercially or may be made by cutting out cardboard circles.
Small cardboard tickets about 1" by 2".
Milk bottle caps which may be called dollars.
Buttons and button molds which are flat on one side.
Toothpicks or small sticks of any kind which may be imagined to be sticks of candy or candy canes.
Small cut-outs of animals, flowers, fruits, which are often available in packages as stickers.
Stickers for special days include witches and pumpkins for Hallowe'en; hearts for Valentine's Day; turkeys for Thanksgiving; trees, bells, candles, wreaths, etc., for Christmas; and chickens, rabbits, eggs, etc., for Easter. Stickers may be used to paste on a large sheet of paper to make picture records of the various number arrangements.
Small-sized clothespins.
Paper dolls.
Small squares of colored paper to represent postage stamps, Christmas seals, or Easter seals.

If scissors and materials are provided, pupils will often be able to make their own cut-outs, their own toy money, or the cardboard tickets.

Large objects which may be moved about at the front of the room or used as stage properties in the dramatization of some life activity include the following:

Objects in use in any classroom—chairs, erasers, books, pads, pencils, pencil boxes, large manila envelopes.
Empty candy boxes, cereal boxes, shoe boxes, which may be used to hold quantities of smaller materials when such materials are being divided into equal groups. They may also be used to represent toy banks, cash registers, pocketbooks.
Paper plates, paper bags.
Blocks, and other toys.
Large clothespins—which are often painted in some color.
Rectangular pieces of cardboard the size of candy bars.
Wooden or paper spoons—which may easily be done up in bundles.
Cardboard tickets about 3" by 4".
Toy money in bills of different denominations—$1, $10, $100 bills. These paper bills may be made by the teacher from heavy paper cut into the correct sizes on the paper cutter and the denominations stamped on with a number stamping set and ink pad. It is advisable to make these bills in different colors; for example, $1 bills green, $10 bills yellow, etc., since the denomination can then be recognized from the color. Shoe boxes appropriately and conspicuously labeled may be used at the front of the room to hold each denomination of bill.

The purposes served in using small objects for individual pupils and in using large objects at the front of the room may differ somewhat. When a pupil uses six of the play pennies on his desk to work out all the possible groupings he can make from six pennies, he is doing his own investigating and making discoveries for himself. The teacher may give him the plus sign as a symbol to use in making number records when he puts his groups together again, but the pupil has worked independently in obtaining the groupings. Likewise, when a teacher asks pupils who need to do so to use pennies to find out how much must be paid altogether for a ball costing 4¢ and a book costing 9¢, each pupil may use his own individual method of obtaining the answer. One pupil may lay out the two groups of pennies and then count 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. Another may count 10, 11, 12, 13. Still another pupil may rearrange his pennies in such a way that he has a group of 10 and a group of 3, and he may then merely think 10 and 3 are 13. Thus materials used individually make it possible for each child to work at his own level of maturity, and to progress naturally from one level to the next higher level. If a pupil already knows that 4¢ and 9¢ are 13¢, he will not bother to lay out pennies at all, but will be ready immediately with the answer.

While individual differences among pupils make it necessary for pupils to work on different levels, it is the obligation of the teacher to guide pupils toward higher levels of thinking. Sensing just when and how to do this is the essence of skill and artistry in teaching.

If classes could be kept small enough so that the teacher could give adequate attention to each pupil, materials for use at the front of the room might not be necessary. However, with large classes it is often desirable to introduce a new concept to the entire class by means of some activity carried on at the front of the room. It then becomes necessary to utilize objective materials which can readily be seen from all parts of the room. We may take as an illustration the building of the number chart to 100, in which the teacher wishes to introduce pupils to the characteristics of our number system, with special emphasis upon the ten group as a basis of comparison for all numbers above 10. Let us suppose that the teacher decides to use clothespins.

The first step is to count out 10 clothespins and record the numbers from 1 through 10 on the board, one number as each clothespin is taken out of a large box. After the tenth one has been added, the teacher may say that she is going to make the group easier to handle by putting a rubber band around all ten. The pupils' attention may then be directed to the following ideas:

a. The bundle or group of 10 may be thought of as one 10 as well as ten 1's.
b. When we *write* 10, we consider it as one 10 because we use a "1," but we give the "1" a new position to the left of the one's place.
c. To show that the "1" is in ten's place, we fill the one's place with a zero.

When one more clothespin is put with the ten group, 11 may be written at the top of the second column. Pointing to the 11 the teacher may ask which 1 indicates the 1 bundle of ten and which 1 stands for the 1 single clothespin. The same procedure may be followed as the numbers 12, 13, etc., through 19 are written. When there are 10 single clothespins again, another bundle of ten may be made and the number 20, which shows two 10's with a zero in the one's place, may be written at the bottom of the second column.

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

At a later time, the number chart to 100 may be completed on the board as concrete materials are counted and grouped into tens and ones. As soon as pupils "catch on" to the system of repeating ten groups and combining ones, they may proceed to complete the chart without the use of objective materials.

After the activity of building a number chart through the use of concrete materials, the teacher is able to focus the pupils' attention upon the fact that the chart presents an organized record of all the different sized groups which have been formed and the way they were formed.

The above activity illustrates the use of objective materials for the purpose of developing what may be termed pure mathematical concepts. Carried through in the manner described, the activity does not serve an immediate social purpose. However, in follow-up activities, the making of ten groups may be shown to have social value. An imaginary social situation such as the following may be dramatized.

> At the school entertainment last night Bob took tickets at one entrance and Jim at the other entrance to the auditorium. This is Bob's box of tickets and this is Jim's. What would be the most accurate way to *count* the tickets in each box?

One box may be counted by ones, and the other by making bundles of tens, then counting by tens and adding the ones.

Pupils will probably recognize the fact that in counting without grouping it is very easy to lose count, particularly when the numbers are large.

A large number of play pennies in a penny bank (an empty cylindrical salt box with a slit cut in the top) may be counted by grouping them into tens. Thus a variety of activities may be used to emphasize the convenience of grouping things into tens when it is necessary to count a large number.

When objective materials are used for demonstration purposes, the teacher always needs to make certain that the pupils go beyond the mere activity and try to sense the *idea* which is being demonstrated. The teacher may stimulate this type of thinking through his questions and by placing emphasis upon certain points. In the organization of a number chart to 100, emphasis is placed upon two points: (1) whenever there are enough ones to group as another ten, this grouping takes place; (2) we use the symbols 1, 2, 3, 4, etc., to tell the number of *ten groups* we have, but we write one of these symbols at the left of another symbol which means ones, and if there are no ones, we use a zero in one's place.

In the same manner, pupils at work with their own objective materials need skillful teacher direction and guidance to help them to make generalizations and to help them to move from immature to more mature ways of thinking about numbers. It should be recognized that the manipulation of concrete objects represents only the first stage in the development of pupils' number ideas. In the second stage of progress, pupils are able to "think" certain arrangements when the objects are present only in imagination, and in the third stage, no objects are present or imagined and pupils have developed the ability to use the language of number. As pupils analyze, assemble, and compare groups of objects, the teacher needs to guide their thinking to the point that advancement will be made steadily toward a higher stage.

Listed among objective materials were bills in denominations of $1, $10, and $100. When it is necessary to extend pupils' number meanings to include numbers from 100 to 1,000, it is often advantageous to use these bills to help some pupils to visualize large numbers. At first ten $1 bills are used together to represent the ten group, then a single $10 bill may be substituted. To represent the hundred group, one hundred $1 bills grouped as ten 10's may be used at first, with later a single $100 bill used. To build the new number group of 1,000, we may then use ten $100 bills since it would be improbable that enough $1 bills would be available.

Large amounts of money, such as $4,235, may be illustrated visually by using forty-two $100 bills, three $10 bills, and five $1 bills. Pupils often fail to sense the difference in value made by omitting a decimal point from numbers which represent money. It has been found very effective to have pupils use toy money and show visually the bills and coins used to indicate amounts of money. For example, pupils are more impressed with the difference in value between $432 and $4.32 after they have used toy money to count out both amounts. For $432, pupils may use four $100 bills, three $10 bills, and two $1 bills, or they may use forty-three $10 bills and two $1 bills. For $4.32, pupils may use four $1 bills, 3 dimes, and 2 pennies.

Using concrete materials in developing meanings and symbolism for the four processes. Activities and objective materials, both for individual pupil use and for classroom demonstration, are particularly important in the development of the meanings of the four processes. The same objective materials suggested above for developing number concepts may be used effectively.

Addition and multiplication are dramatized as combining or "putting together" procedures while subtraction and division are shown to be separating or "taking apart" processes. The symbols which facilitate the recording of these arrangements must be linked very closely with the activities and the thinking involved. While concrete objects and pupil activity are important in the introductory phase of each operation, pupils should be encouraged as soon as possible to imagine the activity indicated by the words in a described social situation or by the sign in an abstract problem. Through early work in building and analyzing numbers, pupils will have gained some acquaintance with the addition and subtraction concepts and also with the language associated with these processes.

A new process and its accompanying new sign may well be introduced through the dramatization of a life situation familiar to the children. Several such dramatizations, all of which necessitate the use of individual pupil materials and activities, are outlined below.

Concept to Be Introduced: Addition

Social situation: Bob brought a dime to school for his lunch. He noticed that a sandwich cost 6¢ and a bottle of milk cost 4¢. Did Bob have enough money to pay for these two things?

Objective materials needed: A box of play pennies for each pupil.

Method: Individual pupil manipulation of play pennies.

T: "Can you show on your desk in one row the number of pennies the sandwich costs, then show in another row the number of pennies the milk costs?" The pupils lay out the pennies thus:

o o o o o o
o o o o

T: "Now find out how many pennies you have laid out altogether." Pupils will use different methods to find out the total number in both rows. When pupils obtain answers the teacher places a number picture of what they did on the board, using the addition sign to indicate that two numbers were put together.

Symbolic representation:

6¢ for the sandwich
+4¢ for the milk
10¢ fo[illegible]h

Explanation: We write a *plus* sign to show that Bob wants to know how much *both* things cost. The amount both things cost is written under the short line drawn.

To further emphasize the combining of two groups, pupils may pick up the six pennies and hold them in one hand, then pick up the four and hold them in the other hand. Then the four pennies may be *put together* with the six pennies, as indicated by the new sign. The number written under the short line shows how many *altogether*.

In discussion of the activity, pupils learn to use the words *both*, *put together*, and *altogether* in conjunction with the activity and the new symbolism.

Concept to Be Introduced: The Additive Idea of Subtraction[*]

Social situation: Each pupil in the room was asked to bring 15¢ for bus fare to the Flower Show. Jack already had 8¢ in his pocket. Jack wondered how many more pennies he needed to bring from home.

Objective materials: A box of play pennies for each child.

Method: Individual pupil manipulation of play pennies.

T: "Show on your desks the number of pennies Jack needs for bus fare." (The pupils lay out 15 pennies in a row.)

T: "Put your hand or a card over the eight of these pennies that Jack already has."

T: "How many pennies are not covered? Can you tell now how many more pennies Jack needs?"
If it seems necessary, the teacher may draw pennies on the board to represent the real pennies, then cover up 8.

Symbolic representation:

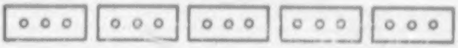
15¢ Jack needs
−8¢ Jack has
———
7¢ more needed by Jack

Explanation: Since the 8 pennies, which represent the group that Jack has, may be covered up or taken away, we may write this as a subtraction example.

Concept to Be Introduced: The Measurement Idea of Division

Social situation: Mary's mother gave her a dime and a nickel and told her to get 3¢ stamps with it. Mary wonders how many stamps she should ask for at the post office.

Objective materials: A box of toy money containing dimes, nickels, and pennies.



Method: Individual pupil manipulation of toy money.

T: "Can you show on your desks the number of pennies for which Mary could exchange her dime and nickel?" (It is presupposed that pupils know the value of a dime and a nickel so pupils will be able to lay out 15 pennies.)

T: "Can you make a pile of pennies to buy one stamp?" (Pupils put 3 pennies in a pile.)

T: "Now see how many piles with 3 in a pile you can make?"
If it seems necessary, the teacher may draw 15 pennies on the board and group them into threes.

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[*] The additive idea of subtraction was selected as one illustration. The take-away and the comparative ideas of subtraction may also be dramatized.

Symbolic representation:

$$\begin{array}{r} 5 \\ 3\overline{)15} \end{array}$$

Explanation: This frame $\overline{)}$ which we put over 15 means that the number under it is to be separated into groups. The 3 we write outside the frame shows how many are to be put into each group. The 5 we write above the frame shows how many groups we are able to make, or, in this instance, how many stamps we can buy.

Concept to Be Introduced: Borrowing in Subtraction

Social situation: Jane got on the street car with 4 dimes in her pocketbook. The carfare was 6¢, which Jane had to put in the box. What did Jane have to ask the conductor to do for her so that she could put the 6¢ in the box?

Materials: Toy money—dimes and pennies.

Method: Individual pupil manipulation.

T: "Can you all show the money Jane had in her pocketbook?" (The pupils lay out 4 dimes.)

T: "How much money is this?" (Teacher writes 40¢ on board.)

T: "Let's suppose that the conductor changed one of Jane's dimes to pennies. You may show how her money looked then." (On the board the teacher may change 40¢ to appear thus, and the pupils may have on their desks 3 dimes and 10 pennies.)

$$\overset{③}{\cancel{4}}\ \overset{⑩}{0}¢$$

T: "Can Jane put 6¢ in the box now? You may all take away 6 penies and put them on the corner of your desk." On the board the teacher may complete the symbolic representation:

$$\begin{array}{rrl} \overset{③}{\cancel{4}} & \overset{⑩}{0}¢ & \text{at first} \\ - & 6¢ & \text{for fare} \\ \hline 3 & 4¢ & \text{left} \end{array}$$

T: "Suppose Bob had 4 dimes and he owed Tom 13¢? How would you write this in numbers?

$$\begin{array}{r} 40¢ \\ -13¢ \\ \hline \end{array}$$

T: Tell what changing Bob would have to do before he could pay Tom what he owed him."

Explanation: When Bob had 4 dimes he had enough money to pay out 1 dime and 3 pennies, but he did not have it in the right coins. He had to change one coin worth 10 pennies into 10 pennies.

In the dramatization of carrying in addition and multiplication, borrowing in subtraction, and in showing the division process, it is necessary to use money only in denominations which are powers of ten; that is, pennies, dimes, $1 bills, $10 bills, and $100 bills, etc. However, in learning to count money and to make change, pupils need to have available for use toy money in nickels, quarters, half dollars, and $5 bills also.

Further illustrations of the use of objective representation of ones, tens, and hundreds will not be given here, since this is adequately covered in the chapter by Thiele.

Using concrete materials in teaching common and decimal fractions and percentage. The teaching of common fractions in a meaningful manner requires the extensive use of concrete and semi-concrete materials. The materials suggested below are grouped according to the phase of the work with fractions for which they seem particularly adapted.

A Fraction as One or More of the Equal Parts of a Unit. Objective materials which may be used to develop the idea of a fractional part of a unit include:

Apples, candy bars, oranges, cookies, etc., which children find it necessary to divide into fractional parts in their everyday experiences.

A yard of ribbon, lace, tape, string, strip of paper, which may first be folded, then cut into a given number of equal parts.

An empty rectangular cereal box or a cylindrical salt box which may be divided horizontally into halves, thirds, or fourths by using a razor blade and cutting the box into equal sections. These may be taken apart and put together again to show parts of a solid.

A glass measuring cup (the kind without sloping sides is preferable) may be filled with colored water to show vividly ⅓, ¼, ½, ⅔, or ¾ of a cup.

Paper pie plates may be cut into fourths, sixths, or eighths, to represent these fractional parts of real pies. The circular paper disks on which bakery cakes are sold may be used in a similar manner.

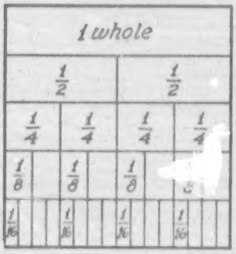
Semi-concrete objects which each pupil may cut into equal parts include rectangular pieces of paper, which may represent candy bars, and circles of tagboard which may represent pies. The parts into which these materials are cut should be appropriately labeled on both sides so that pupils may continually associate the symbol with the size of the piece which is being considered. The circles should all be the same size so that pupils may readily compare the size of certain unit fractions, as 1/2, 1/3, 1/4, and

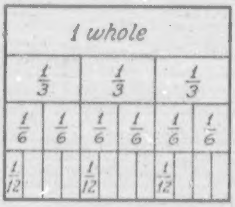
1/8. However, it is helpful for future work in adding and subtracting fractions to have the circles of different colors. Red circles may be cut into halves, blue ones into fourths, green ones into eighths, etc.

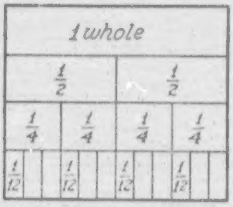
After concrete and semi-concrete materials have been divided into fractional parts, pictures showing the various divisions may be placed on the board for analysis and study. Fractional parts, such as 1/2, 1/4, 1/8, and 1/16, of circles which are the same size may be shaded different colors. As relationships between these parts are discovered by the pupils, the teacher may use symbols to record these relationships on the board.

As rectangles are folded or cut into equal parts, diagrams showing clearly the equivalence of certain "families" of fractions may be placed on the board and also in pupils' notebooks. Each part of a diagram should be drawn on the board as pupils perform the action which the diagram pictures.

A ready-made diagram or chart presented to pupils without accompanying activity is not as effective as one which is built up step by step as pupils use concrete materials. However, many teachers make carefully constructed fraction charts to be shown to pupils *after* the development lesson. These charts are put up about the room where pupils may refer to them when necessary. The following are illustrative.





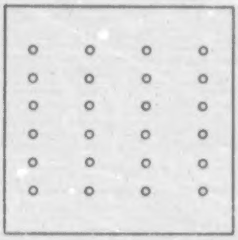


A Fraction as One or More of the Equal Parts of a Group. The objective materials usable before the class include chairs, books, pads, pencils, paper plates, which may easily be arranged in equal groups. The materials usable by each individual pupil include play pennies, milk bottle caps, buttons, small cut-outs, small tickets, etc.

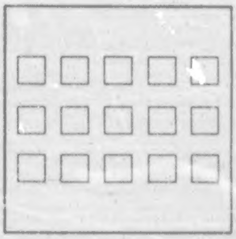
Pupils may show with pennies that 1/4 of 12 pennies = 3 pennies; they may draw 12 pennies in 4 equal groups to picture the activity and they may use only the symbols 1/4 of 12 = 3.

When pupils have used concrete materials to show fractional parts of groups, they have a background for the use of partition division as a second way of expressing the division facts.

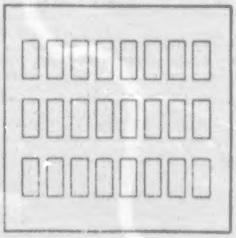
Pictures of circles, crosses, lines, squares, etc., arranged to show fractional parts of groups, are helpful to pupils *after* actual experience in using real objects. Pictures such as the following may be placed on the board. Pupils may use colored chalk and color enough to show each of the relationships given under the picture.



Color:
¼ of all the circles red
½ of all the circles blue



Color:
⅕ of all the squares blue
⅖ of all the squares yellow



Color:
⅓ of all the oblongs green
⅔ of all the oblongs red

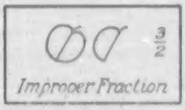
Using Fractions to Express Comparisons. The objective materials listed previously may be used to introduce the idea of using fractions to express comparisons. Situations such as the following may be illustrated with concrete materials.

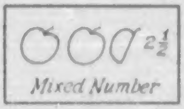
> On one shelf of the bookcase there are 8 readers and 4 arithmetics. There are ½ as many arithmetics as readers, or there are twice as many readers as arithmetics.
>
> In the first row there are 6 boys and 4 girls. There are ⅔ as many girls as boys or 1½ times as many boys as girls.
>
> Also 6/10 or ⅗ of all the pupils in the row are boys while 4/10 or ⅖ of the pupils are girls.

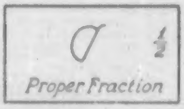
It should be noted that the concept of fractions represented by the last two problems is presented somewhat later in the pupils' school experience than the other fraction ideas.

Illustrating Meaning of Proper Fraction, Improper Fraction, and Mixed Number with Real Objects and Pictures. From the large variety of objects which have been cut into fractional parts and which are available in the room, pupils may be asked to hold before the class materials to illustrate the numbers given below. As each is displayed and as the symbols are studied, the distinction between proper fractions, improper fractions, and mixed numbers may be pointed out.

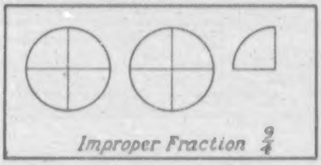
$\frac{3}{4}$ of an apple	$\frac{1}{2}$ of a yard of ribbon
$1\frac{1}{2}$ pies (use paper pie plates)	$\frac{3}{2}$ yards of ribbon
$\frac{4}{3}$ cakes (use cardboard circles)	$2\frac{1}{4}$ yards of tape

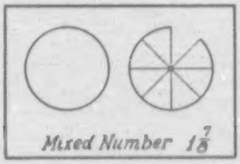


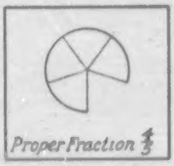




Pictures may also be employed to illustrate the meaning of proper fractions, improper fractions, and mixed numbers. Cut-outs of apples, oranges, bananas, or circles may be used. Some teachers paste these cut-outs on cards and make permanent illustrations which pupils refer to when in doubt as to the meaning of any of these terms.







Illustrating Addition and Subtraction of Fractions and Mixed Numbers with Concrete Objects and Diagrams. The many circles which pupils have cut into halves, fourths, eighths, thirds, sixths, twelfths, etc., may well be placed in a series of boxes, each labeled to show the fractional part it contains. It is suggested that the circles cut up should all be the same size but of different colors.

Many pupils will require a minimum amount of objective or visual representation of addition and subtraction, while other pupils will need a great deal of this type of work. Pupils who need to do so may get materials to show combinations such as the following:

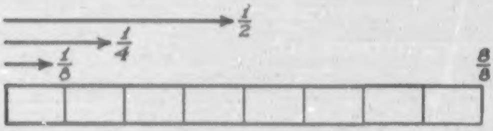
$$\frac{1}{4}+\frac{1}{4} \qquad \frac{3}{4}+\frac{1}{2} \qquad \frac{3}{4}-\frac{1}{4} \qquad 1\frac{1}{2}-\frac{1}{4} \qquad 1-\frac{7}{8}$$

Pupils may be given directions such as those listed below as an exercise in showing addition and subtraction with diagrams.

Directions to Pupils:

Draw 3 diagrams like the one shown on your papers. On one diagram shade ¼ blue. Next to the part you shaded blue, shade ⅛ yellow. What part is shaded altogether? What part is not shaded?

$$\frac{1}{4}+\frac{1}{8}=?$$



On another diagram, shade ½ green. Next to the part you shaded green, shade ⅜ red. What part is shaded altogether? What part is not shaded?

$$\frac{1}{2}+\frac{3}{8}=?$$

On another diagram shade ⅜ orange. Next to the part you shaded orange, shade ¼ green. What part is shaded altogether? What part is not shaded?

$$\frac{3}{8}+\frac{1}{4}=?$$

The addition and subtraction of fractions and mixed numbers which utilize objective and visual aids such as those described above may well be undertaken by children a full year before similar work without such aids is attempted. In whatever grade the operations with fractions are introduced, the first steps should include every opportunity for pupils to use objective materials freely. As pupils gain familiarity and a sense of security in dealing with fractions, they will cease to depend upon visual aids.

Teaching Decimal Fractions. If pupils have clearly developed concepts of common fractions, it is not necessary to employ concrete materials very extensively in introducing the decimal fraction. However, teachers sometimes find it advantageous to continue to use circles and rectangles which may be cut up or pictured to help pupils to visualize tenths and hundredths. The equivalence of certain common and decimal fractions may often be emphasized by the use of diagrams. The following are suggestive:

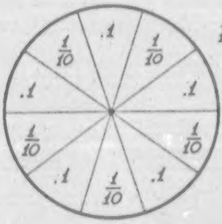


FIGURE 1

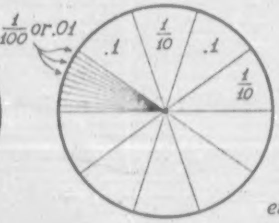


FIGURE 2

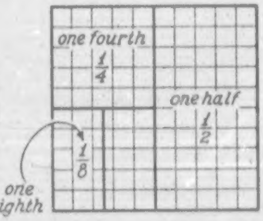


FIGURE 3

Figure 1 presents visually the following relationships:

$$1=\frac{10}{10} \qquad \frac{1}{2}=\frac{5}{10} \text{ or } .5 \qquad \frac{1}{5}=\frac{2}{10} \text{ or } .2$$

Figure 2, although not completed, shows that $1=\dfrac{100}{100}$

$$\frac{1}{10}=\frac{10}{100} \text{ or } .10 \qquad \frac{3}{10}=\frac{30}{100} \text{ or } .30 \qquad \frac{5}{10}=\frac{50}{100} \text{ or } .50, \text{ etc.}$$

Figure 3 may be used to explain the following relationships:

$$1=\frac{100}{100} \text{ or } \frac{10}{10}$$

$$\frac{1}{2}=\frac{5}{10} \text{ or } .5$$

$$\frac{1}{2}=\frac{50}{100} \text{ or } .50$$

$$\frac{1}{4}=\frac{25}{100} \text{ or } .25$$

$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$\frac{3}{4}=.50+.25=.75$$

$$\frac{1}{8}=\frac{12\frac{1}{2}}{100} \text{ or } .12\tfrac{1}{2}$$

$$\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$$

$$\frac{3}{8}=.25+.12\tfrac{1}{2}=.37\tfrac{1}{2}$$

$$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$$

$$\frac{5}{8}=.50+.12\tfrac{1}{2}=.62\tfrac{1}{2}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$

Draw a diagram like Figure 4 of a road one mile long. Divide it into eighths. On the upper scale show the divisions by using common fractions. On the lower scale, show the divisions by using decimal fractions. This device often aids pupils in learning the decimal equivalents for halves, fourths, and eighths.

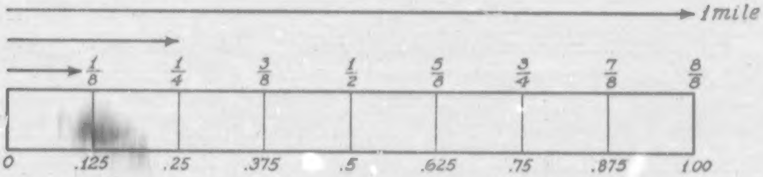


FIGURE 4

Draw a diagram like Figure 5 of a road one mile long. Divide it into tenths. On the upper scale show the divisions by using common fractions. On the lower scale show the divisions by using decimal fractions. This device often aids pupils in learning the decimal equivalents for halves, fifths, and tenths.

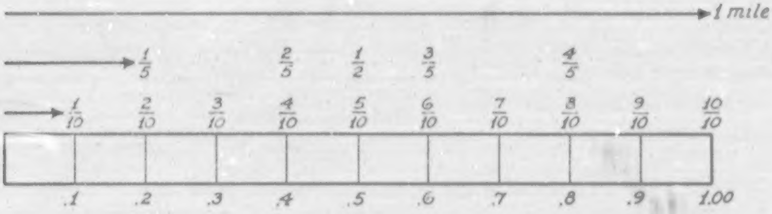


FIGURE 5

Several small pieces of squared paper, each piece containing 100 squares, may be provided for pupils. Pupils may paste them in their notebooks after they have colored parts as directed. Pupils should understand that each piece of squared paper represents *one* unit.

Pupils may use one piece of squared paper to show the following relationships:

a. Shade 3 squares in row A blue. Write a decimal fraction and a common fraction to show what part of the 100 squares are shaded blue.

b. Shade 10 squares in row B brown. Express the part of all the squares which are colored brown decimally in two ways and fractionally in two ways.

c. Use rows C and D and shade enough small squares in red to show that $.11 = .1 + .01$.

d. Use rows E, F, and G, and shade enough small squares in green to show that $.25 = .2 + .05$.

e. Use rows H, I, and J, and shade enough small squares in yellow to show that $.3 = .30$.

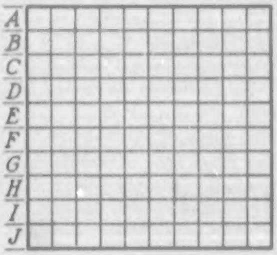


FIGURE 6

The teacher may prefer to draw 100 squares on the blackboard and carry through the above exercise with colored chalk at the board before or instead of having each pupil engage in the activity individually.

Pupils may use another piece of squared paper to show the addition of tenths and hundredths.

a. Use row A. Shade .06 red and .04 blue to show that $.06 + .04 = .10$ or .1. Express this problem fractionally.

b. Use row C. Shade .07 green and .03 yellow to show that $.07 + .03 = .10$ or .1. Express this problem fractionally.

c. Use rows E and F. Shade .08 red and .06 black to show that $.08 + .06 = .14$ or .1 and .04. To do this you will need to fill all the squares in row E before you shade any in row F. Express this problem fractionally.

d. Use rows H and I. Shade .14 green and .06 yellow to show that $.14 + .06 = .20$ or .2. Express this problem fractionally.

Representing Mixed Decimals Visually. Pupils who understand the characteristics of the whole number system will probably encounter little difficulty in gaining the concepts of pure decimals and mixed decimals. However, teachers who have found work in picturing pure decimals, as described above, helpful for slow pupils, often continue the work with concrete materials to show mixed decimals in some such manner as the following:

One piece of squared paper, containing 100 small squares, is used to represent a unit; one strip or 10 small squares represents one tenth; and one tiny square represents one hundredth. Boxes, appropriately labeled, are provided to hold a supply of units, tenths, and hundredths.

Exercises in building small mixed decimals concretely are then given. To build 1.23, the pupil may get 1 large square which represents 1 unit, 2 strips which represent .1 each, and 3 tiny squares which represent .01 each. These parts may be pasted on a piece of paper of another color to represent visually the mixed decimal 1.23.

Teaching Percentage. Percentage presents no new difficulties for pupils who have a thorough understanding of common and decimal fractions. All the concrete materials and exercises listed for use in teaching hundredths may be applied to the teaching of percentage, if this seems necessary, by merely changing the terminology from hundredths to per cents.

Utilizing objective materials in developing concepts of measures. Pupils develop understandings of units of measure and skill in the techniques of measurement by examining and using measuring instruments. Pupils may obtain some general ideas from observing the teacher or other pupils carry through measurement activities, but the greatest value comes from the actual use of measuring instruments by each individual pupil. Suggestions for objective materials for the different types of measurement are given below.

Linear Measure

1. Measuring instruments:
   Foot rulers, yardsticks, yard tape measures, 6 ft. carpenter's rule or 6 ft. steel tape, a 25 or 50 ft. steel or cloth tape measure.
2. Objects which may be measured by pupils *after they have estimated* the measurements to be taken:
   a. For measurement in inches—
      Books, blotters, notebooks, desks, erasers, pencils, cereal boxes, tin cans.
      The dimensions of No. 2 and No. 2½ sized tin cans may be compared.

The dimensions of 1 lb., 2 lb., and 5 lb. candy boxes may be compared.

b. For measurement in feet—

Height of pupils in room.

Length and width of blackboards, windows, doors, maps, table tops, bulletin boards.

Height of tables, desks, chairs, window sills, door knob.

Height of shelf from floor, height pupils can reach.

Measurements in the gymnasium or out-of-doors may include: laying off distances for 50 and 100 yard dash, laying out baseball diamond, tennis court, volleyball court.

c. For measurement in yards—

Materials usually sold by the yard. The pupils and teacher may bring to school actual materials or something to represent the materials sold by the yard:

Used Christmas ribbons.
Used lace.
Cellophane strips, such as new lamp shades are wrapped in.
Shelf paper and edging for shelves.
Ball of string, spool of thread.
Roll of wrapping paper.

d. For measurements in miles—the importance of the car speedometer—

To gain the concept of a mile, pupils need to locate a place which is approximately a mile from the school, then cover this distance by walking, by riding a bicycle, or by riding in a car. In a car the trip speedometer should be set at zero or at an integral number of miles so that pupils may note where they are when they are .1, .2, .5, and .9 of a mile from the starting point.

If the length of the school yard has been determined, pupils may find out how many times this distance they would need to walk to go a mile. Finding the average length of the blocks in the neighborhood, then finding the number of blocks in a mile, is also helpful.

Some help may be gained if a city map is available showing ½ mile, 1 mile, 2 mile circles, etc.

3. Construction of foot rulers and tape measures by the pupils:

If it is impossible to provide as many foot rulers and yardsticks as are necessary, pupils may make their own foot rulers and tape measures. Pupils may bring from home discarded cardboard suit boxes or the large pieces of cardboard put in men's shirts by the laundries. From these the teacher may cut on the paper cutter strips 12 inches long and 1 inch wide. Each pupil may be provided with a piece of stiff paper 1 inch long which he may use as a unit to divide the 12 inch strip into inches. The 1 inch unit may next be folded into halves and used to mark off the half inches on the ruler.

A strip of cloth tape 1 yard long may likewise be marked off into inches and fractional parts of inches to make a tape measure.

Measures of Weight

1. Measuring instruments:
    - Several pairs of kitchen scales, weighing up to 25 lb.
    - Postal scales for weighing accurately in ounces and fractions of an ounce.
2. Articles available in the classroom whose weights may be estimated, then checked by weighing: books, notebooks, pencil box, blackboard eraser, box of scissors, flowerpots, etc.
3. Articles which the teachers and pupils may bring in to weigh:
    - Dry beans which may be sewed into cloth bags in 1 lb. quantities and used for pupils to lift so they get the "feel" of 1 lb., and have a basis for estimating other weights.
    - Foods which are sold by weight—potatoes, apples, squash, etc., which are not too perishable.
    - Boxed or canned goods, the net weights of which are given on the labels, and the weights of which pupils may check.
    - Empty boxes and cans which may be weighed.

Measures of Time

An alarm clock and a stop watch.
A watch or clock with a second hand.
A cardboard clock, consisting of a clock face drawn on cardboard and supplied with movable hands.
A series of clock faces printed to show various times to provide practice in telling the time.
A series of blank clock faces printed on paper. Pupils merely insert the hands to show designated times.
An egg timer proves interesting to pupils.

Liquid Measure

1. Measuring instruments:
    - Glass measuring cup marked into thirds, fourths, halves.
    - Separate measuring cups holding ⅓, ¼, and ½ cup each.
    - Pint, quart, and 2 quart fruit jars.
    - ½ pint, pint, and quart glass milk bottles.
    - Pint and quart milk cartons.
    - Glass gallon jug.
    - Gallon tin can.
    - ½ and 1 gallon paint cans.

Dry Measure

Pint and quart berry box.
Peck, half bushel, and bushel baskets.

Square Measure

1. The concept of area—developed through the use of square inches:

Pieces of cardboard 1 inch on a side may be provided for each pupil. Small rectangular pieces of paper an exact number of inches on a side, a different sized piece for each row, may also be provided. Row I's pieces may be 3″ x 2″; row II's 2″ x 4″; row III's 3″ x 5″, etc. Each pupil may be directed to use his square of cardboard as a unit of measure and by actually applying it, mark off the number of square inches along the length of his small rectangular piece of paper. He may then find out how many rows of square inches there are altogether.

If the pupil cannot tell the total number of square inches at this point, he continues to mark off all the squares. Following one or more experiences of this type the teacher may guide the pupils in the formulation of their own generalization regarding the method of finding areas in square inches.

2. The concept of a square foot:

A piece of cardboard 1 foot on a side may be used in a similar manner at the blackboard to find the area of a figure 3 ft. by 4 ft. Thus it may be demonstrated that the generalization applies to finding areas in square feet as well as in square inches.

It may be demonstrated that 1 square foot can be divided into 144 square inches by drawing lines at the blackboard or more vividly by actually cutting up 1 sq. ft. of paper into 144 sq. in. Fold a piece of paper 1 ft. on a side into 12 strips, then cut them apart. Mark off one strip into 12 parts, clip together the strips, and cut off the 1 inch squares. One demonstration at the front of the class is sufficient to impress the pupils with the fact that 12 times 12 square inches can be cut from 1 square foot.

3. The concept of a square yard:

Pupils may mark off several spaces on the floor which are 1 yard on a side. Each one may then be divided into 9 square feet.

It is sometimes possible to have a piece of something which is 1 yard on a side, as, a square yard of linoleum, carpet, corkboard, or cardboard.

4. Finding areas of rectangles:

Objects in the classroom having rectangular surfaces of which the area may be found include the following:

Area of glass on pictures, panes of glass in window and door
Areas of desks, tables, sections of the blackboard
Area of bulletin board, of posters, of charts
Find and compare areas of different classrooms.

5. Development of the formulas for areas of parallelograms and triangles:

a. Area of a parallelogram—

Cut a parallelogram out of paper or cardboard such as ABCD (Figure 1). From A, drop a perpendicular to CD. Cut off the triangle formed at the left and fit it on to the right side of the figure as in Figure 2. It may be fastened on by using transparent cellulose mending tape.

Since the area of the rectangle formed in Figure 2 is found by multiplying the base by the height, the area of the parallelogram is found by the same rule.

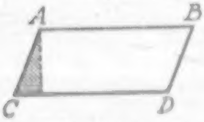


FIGURE 1

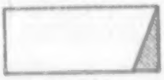


FIGURE 2

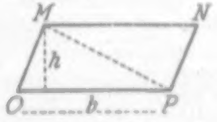


FIGURE 3

b. Area of a triangle—

Cut another parallelogram from paper or cardboard such as MNOP (Figure 3). Show the height (h). The area of the parallelogram = bh. Draw the diagonal MP of the parallelogram and cut along this line. Since the two triangles we have formed can be shown to be equal, the formula for the triangle is one half the base times the height.

6. Use of squared paper in finding areas:

Figures such as rectangles, parallelograms, triangles, trapezoids, and irregular shaped figures may be drawn to scale on squared paper. A drawing of this type provides a method of visualizing clearly the number of square units in the surface represented and often a close approximation can be obtained from the scale drawing.

Cubic Measure

In developing the concept of volume as the number of cubic units of a given kind that a solid contains, it is helpful to have available a supply of tiny cubes. Sometimes cubes of loaf sugar are used to build up small solids, which can then be shown to contain as many cubic units as there are units in one layer times the number of layers.

Cubes of wood, soap, clay, or paper may be used in a similar manner.

In work involving the finding of volumes of different solids, it is important that pupils have clear mental pictures of the solids under consideration. For this purpose a variety of objective materials may be used. The following list is merely suggestive:

Models of cubes, prisms, cylinders, cones, pyramids, spheres, and hemispheres made of wood, soap, clay, or paper.

Familiar objects in the environment which are made in the above shapes and which may be brought into the classroom for measurement and finding of volumes.

Rectangular solids—candy boxes, cereal boxes, shoe boxes, butter cartons, boxes frozen vegetables come in.
Cylindrical solids—tin cans used for fruits and vegetables, cylindrical salt boxes, hatboxes, a stove pipe, a water pipe, some waste baskets, pencils, some flower containers, kettles, fruit jars, pans, some cake tins, casseroles, pillars, pails.
Cones—ice cream cones.
Spheres—children's rubber balls, baseballs, golf balls, grapes, oranges, grapefruit.

Intuitive Geometry

The intuitive geometry work in grades seven and eight includes work with lines, angles, plane figures, and solids. In addition to the materials suggested for use in studying areas and volumes, the following may be listed:

Measuring instruments—
  Meter stick
  Compasses
  Protractor
  Triangles and T squares
  Transit
  Plumb-bob
  Carpenter's level

Pictures and illustrative materials—
  Pictures which show geometric forms in architecture and in design.
  Samples of linoleum, tapestries, wallpaper, cloth, which use geometric forms and designs.
  Collection of buttons illustrating a variety of geometric forms.

There remains another important aspect of measurement which has been merely mentioned in a few instances—the realization of the importance of units of measure in carrying on the business of the community. In the actual teaching of measurement, no separation should exist between the study and use of measuring instruments and the social uses of the various units of measure. For the purposes of this chapter, however, it seemed better to discuss these two aspects separately. The social applications are included in Part II of this chapter.

II. UTILIZING SUPPLEMENTARY MATERIALS AND ACTIVITIES IN GIVING SOCIAL SIGNIFICANCE TO ARITHMETIC

To make arithmetic truly significant for pupils, teachers find it necessary to study carefully the ways in which arithmetic functions in the home, school, and community, and then devise methods of helping pupils to develop an awareness of the large variety of life situations in which arithmetic is used, and methods of helping pupils to deal successfully with the quantitative aspects of the situations discovered.

The arithmetic textbook seeks to emphasize the significance of numbers by including a great variety of problems to solve, problems of the type that arise in the everyday experiences of the children or their parents, or problems which the children may be called upon to solve in later life. The problems may be of two types: a series of problems all based upon a single social situation, or a group of problems about different social situations, but all emphasizing the uses of some one phase of the work, as, fractions, decimals, or percentage. The success of textbook problems in giving significance to arithmetic depends upon the extent to which pupils recognize them as representative of real life situations.

Teachers often find that added significance may be given to arithmetic if the problems in the textbook are supplemented by certain types of activities which are selected with a view to the opportunities which they provide for understanding and using numbers. Under this classification we may include the following:

- Dramatization of life situations and business procedures.
- Interviewing experts to obtain information about topics being studied.
- Taking trips.
- Preparation of posters, bulletin board displays, and other types of exhibits.
- Making a class scrapbook or individual scrapbooks to hold clippings, illustrations, and other supplementary materials.
- Preparation and giving of special reports.
- Construction of measuring instruments, of models, and of various devices.
- Analyzing imaginary life situations having quantitative aspects about which first-hand information may be obtained.
- Participating in activities of the home, school, and community which have definite quantitative aspects.

Illustrations of the manner in which some of these activities may be initiated and carried through are given below. It will be noted that often the use of one type of activity leads very naturally to the use of several others.

Using dramatization of store activity in teaching whole numbers and measures. The motivation for the study of the combinations and the four operations with whole numbers is often accomplished by the dramatization in the classroom of some kind of store activity. Careful guidance by the teacher is necessary in helping pupils to decide upon the kind of store they wish to set up. The choice will, of course, depend upon the grade level and the abilities of the pupils, and upon their familiarity with a similar kind of store in their out-of-school experiences. To reproduce in the classroom any type of store activity presents three requirements: (1) Certain materials to serve as stage properties, (2) definite information obtained from the real situation, (3) certain mathematical abilities.

The grocery store is one of the types of stores most commonly selected for dramatization. A corner of the classroom may be transformed into a more or less realistic store, depending upon the number and type of stage properties which children and teacher contribute.

Socially valuable outcomes from the preparation of a grocery store and the dramatization of buying and selling in the store include the following:

By reading price lists from different stores, pupils become aware of the range in price which exists for certain articles.

Pupils realize that they need to find an average price to charge for the articles included in their store.

Pupils find that fixed prices exist for certain standard brands of goods.

Pupils come to realize the effect upon prices of large quantity buying, of in-season buying of fruits and vegetables, of buying at special sales, etc.

Pupils learn the unit in which each article is sold, learn the meaning of the term "net weight," and come to realize the importance of studying the information contained on the label of packaged, canned, and wrapped goods.

Pupils learn how to weigh things.

Pupils realize the necessity for learning to estimate the total amount of their bill, or the amount of change they will receive, as well as learning the method of ascertaining the exact total and the exact amount of the change.

Pupils learn to count money and become familiar with the procedure of making change.

By carrying on buying and selling transactions, pupils learn to use the four operations with whole numbers.

Other types of stores about which pupils may gather information preparatory to carrying on a dramatization include the following:

Grades 1-2-3

- School supply store
- Fruit and vegetable stand
- Ten-cent store
- Five-cent to $1.00 store
- Post office
- Cafeteria
- A bakery

Grades 4-5-6

- A florist shop
- A jewelry store
- A hardware store
- An electrical appliance store
- A clothing store
- A furniture store

As pupils visit stores and read advertisements to obtain exact information, they become aware of the important part played by standard units of measure in the transaction of business. Some teachers have pupils accumulate in their notebooks illustrations of the units of measure used in each type of store investigated.

The great variety of uses for different measures in a hardware store is easily understood when each pupil assumes the responsibility for making a classified list such as that given below.

Goods Advertised by Hardware Store

Uses for liquid measure:
  House paint—$2.39 a gallon. A gallon covers 600 sq. ft.
  One coat enamel—59¢ per quart
  Liquid wax—1⅓ pt. can for 59¢; ½ gal. can $1.29; 1 gal. can $2.39
  Spray for shrubs—1 qt. can 65¢; 1 gal. can $1.55
  Furniture polish—$1.29 per quart
  Saucepans with covers, 3 quart size—79¢

Uses for measures of weight:
  Paint cleaner—4 lb. for $1.00
  Oil soap and wool sponge—3¾ lb. can $1.10
  Grass seed—29¢ for 1 lb.; 79¢ for 3 lb.; $1.25 for 5 lb.
  Grass seed—40¢ for 1 lb.; $1.10 for 3 lb.; $1.75 for 5 lb.

Uses for linear measure:
  Electric ¼ in. drill—$7.95
  Window shades 3′ x 6′ as low as 79¢
  100% pure manila rope, waterproof, ¼″ thick—100 ft. for 45¢
  Clotheslines—size 7/32″ diameter—100 ft. for 29¢
  Door mats, 14 by 24 inches—$1.00
  42″ sink, cabinet type—$29.95
  Dish cabinets, white enameled, 68″ high—$5.88
  Gas stove, 16-inch oven fits into space 36 x 28 inches—$36.95

Refrigerator pan, 14 x 8¾ x 5″ size—88¢
Stepladder, 4 ft. size—98¢
Garden hoe, 6½ inch blade, 4½ ft. ash handle—69¢
Lawn mower, 10 inch wheels, 16 inch cut—$5.55
Fencing, 42″ high, galvanized 11 gauge wire fencing—7¢ per lineal foot

Uses of square and cubic measures:

Venetian blinds—35¢ per square foot
Inlaid linoleum—$1.25 per square yard
Refrigerator—6 cu. ft. size—$110.98

Dramatization of the operation of a bank. Pupils in grades seven and eight, as well as younger pupils, profit greatly from the dramatization of life situations in the classroom. In preparation for the dramatization of a bank, pupils may visit a bank or interview some one familiar with local banking practices to find out:

The titles and duties of the men who work in a bank.
The procedure in opening a savings account or a commercial account.
How much money it takes to open a commercial or a savings account.
How to make out deposit slips, withdrawal slips, and requests for cashier's check.
How to endorse and cash a check.
Why it is necessary to be identified when asking to have a check cashed.
The rate of interest paid on savings accounts.
The method used for computing interest on savings accounts.
The charges made by the bank for carrying a commercial account.
The procedure in borrowing money from a bank.
The rate of interest charged by the bank for the loans it makes.
What is meant by collateral and why it is required.
What goes on in a bank before the opening hour and after the closing hour.
That the bank often acts as collection agent for gas, telephone, and electric light bills.
That the bank has safety deposit boxes where valuables may be kept.
That the United States insures bank deposits up to $5,000.

In the play-bank opened in the classroom, all the activities of a real bank may be carried on if the following materials are provided:

Small notebooks to serve as bank books.
Deposit and withdrawal slips, which may be mimeographed forms made like those used in the bank visited.
Bank statement—at least one which pupils may examine.
Blank checks and stubs.
Promissory notes.
Toy money—both coins and bills—and a cash drawer.

As work with the activity progresses, a bulletin board exhibit may be prepared showing the forms used in a bank; a diagram may be drawn to show the travels of a check from the time it is written until the canceled check is again returned to the one who made it out; an interest table may be exhibited.

The study of banking may include the United State Postal Savings System, and the bulletin board exhibit may show samples of postal savings stamps, postal savings certificates, and pamphlets distributed by the post offices which give information about postal savings bonds.

Similar types of material may be collected by each pupil and placed in his notebook.

Organization and operation of a stock company. The following activity was actually carried on very successfully by the eighth grade pupils in one elementary school for several successive semesters.

The pupils decided to organize a company for the purpose of putting out a monthly school paper in mimeographed form. A charter to do business was obtained from the principal. The amount of money required to start the business was determined and was raised through the sale of shares at 10¢ each to members of the class and to teachers in the school. Stock certificates were mimeographed, filled out, and given to the stockholders of the company.

As the business of mimeographing and selling the papers continued, careful account of the expenses and the income from the sale of papers was kept. When the profits warranted it, a dividend was declared and distributed.

Using original arithmetic plays. Gifted pupils may often be stimulated to write and produce plays which use information gained in the arithmetic class. Opportunities of this type provide for an enriched program for the gifted pupils, and when the plays are given for the entire class they often serve to clarify concepts for the less gifted pupils. The following titles for plays are suggestive:

When the Life Insurance Salesman Called
The Adams Family Decides to Budget
To Buy—or Not to Buy—on the Installment Plan

The outline of scenes below indicates how one business transaction, involving the payment of commissions, may be made the basis for a play.

A Real Estate Salesman Believes in Signs

Scene I—The real estate man is in his office. He receives several telephone calls from people who ask him to help them rent or sell their property. He writes some ads which he telephones in to the paper.

Scene II—He drives out to look at the property and places "For Rent" signs on some and "For Sale" signs on others. He stays at the house having a "For Rent" sign.

Scene III—Several prospective renters come into the house. He tells them the price and shows them around the property, telling them the advantages of living there. Finally one of the lookers says that he will rent the house. The real estate man then asks him for references and asks him to read and sign a form which shows the conditions under which the place is rented. The renter pays a month's rent in advance and receives a receipt for it.

Scene IV—The owner of the property is in the real estate office and the real estate man is telling him that he has looked up the references of the man who wishes to rent his place, that they are satisfactory, etc. The owner then receives the balance left from the first month's rent after the real estate man takes out 33⅓% for his work.

Using trips and interviews with experts as supplementary activities. If pupils are to dramatize life situations in the classroom, they need all the up-to-the-minute information about the situation which the community can provide. Trips by individuals, by committees, or by the entire class to certain places in the community provide means of stimulating interest, of giving first-hand experiences, and of obtaining authentic information. Careful preparation should be made by the teacher before the trip so that pupils are aware of definite observations to make and of specific types of information to obtain. Places which may be visited with profit include: stores of different kinds, a wholesale market, a bank, the stock exchange, the post office, the office of some public utility—gas, electricity, water, telephone.

Trips away from the school are sometimes difficult to arrange. Exact information can often be obtained by pupils by consulting their parents or by interviewing friends of their parents. It is also sometimes possible to arrange to have a merchant, a banker, or a salesman come to the school to discuss certain aspects of business with the pupils and answer questions. Experts in their fields may also be able to provide pupils with bulletins, booklets, and pamphlets which may be studied in the clossroom.

Preparation of posters, bulletin board displays, and exhibits. In many classrooms one bulletin board is kept exclusively for clippings, posters, and other types of information related to arithmetic. During the study of each topic pupils display materials which indicate the application of the topic to as many out-of-school situations as possible.

Bulletin Board Display Materials for Decimal Fractions

Daily weather reports from the newspaper showing that the rainfall throughout the country is stated to the nearest hundredth of an inch.

Portion of the sport page—showing batting averages as three-place decimals.

Clippings giving speed records which use one- and two-place decimals in stating parts of a mile, parts of a minute or a second.

Maps and travel information in which distances are given correct to the nearest tenth of a mile.

A drawing of a clinical thermometer, showing its gradation into tenths of degrees.

A drawing of an automobile speedometer showing how tenths of miles are indicated on it.

Bulletin Board Display Materials for Percentage

Advertisements showing the per cent of discount allowed during sales of different kinds.

Schedules showing rate of interest charged by small loan companies, banks, and credit unions.

Newspaper articles which use per cents to indicate comparisons in business conditions, in housing conditions, in health statistics, in records of accidents, in sport records, in population trends, etc.

Budget tables showing per cent of income to be spent for each item in the budget for different incomes.

School records showing per cent of absence, of tardiness, of illness, and also showing per cent of children receiving the different school marks.

Analysis of the local tax budget showing per cent of income allowed for the different departments of government—schools, public works, police department, etc.

Exhibits Other than Bulletin Board Displays

Measuring instruments—which may often be borrowed from various sources, such as a light meter, pedometer or speedometer, transit, micrometer, carpenter's level, a plumb-bob, meter stick, etc.

Collections—coins from different countries; collection of buttons illustrating a variety of geometric shapes.

Analyzing imaginary life situations which have quantitative aspects. Another type of supplementary activity suggested was that of studying some imaginary life situation for the purpose of answering certain important questions with regard to it in the light of information obtained from the community. Several illustrative situations are discussed below.

Proponents of the "incidental learning" theory would not include in the curriculum any arithmetic except that which occurs naturally in some integrated unit which is being studied. The situations described below may appear similar to the units of work used in an activity curriculum; however, they are suggested here because they provide one of several means of helping pupils to use a definite body of arithmetic knowledge and skills which are to be acquired.

SITUATION

The Adams family, which consists of the mother, father, and two children, are planning a summer vacation trip to San Francisco to visit the World's Fair. They plan to drive out in their Ford car. They wonder how much money they need for the trip.

Questions to Be Decided

What possible routes are there for going out and for returning?
What is the mileage by each route?
What points of interest are there en route which should be visited?
Decide approximate distance to plan to drive each day so that places to stop overnight may be located.
How many days will be taken for the trip out and back, including stop-overs at places en route?
How many days should be allowed in San Francisco and vicinity?

How much will the car expenses be for the trip—cost of gasoline, oil, lubrication?

What will the cost of food and lodging be for the family for the trip
(*a*) if stops are made at hotels?
(*b*) if stops are made at cabins?

What will the expenses be while in San Francisco attending the Fair—hotel expenses, food, sight-seeing?

What repairs will the car need before starting out, and what will they cost?

Should extra insurance be carried? What kind? How much will it cost?

Obtaining the Information Needed

Obtain road maps and mileage charts from gas stations or local automobile clubs.

Obtain books showing cabins and hotel accommodations at the places where overnight or longer stops are to be made.

Obtain booklets and pamphlets describing points of interest en route, as well as those describing things to do and see while in San Francisco.

Write for information about prices of gasoline in different parts of the country, and find an average price per gallon.

Obtain information from a Ford car owner about the approximate number of miles the Ford car will go on one gallon of gasoline. Also, find out how often oil will probably have to be added, and how often oil needs to be changed.

Variations of Situation—Planning a Trip

Taking the train instead of driving. If the trip is to be made by train, pupils will need to get timetables and information about:

  Cost of tickets—first-class, tourist, and coach rates.

  Cost of adult and children's tickets for each class of fare.

  Cost of berths and meals for each class of fare.

  Time different trains leave and which ones make best connections if stop-overs are planned.

Traveling out to California by boat through the Panama Canal—back by train:

  Information about fares on different steamship lines, about length of time trip would take, etc., will need to be obtained.

  Plan the trip for 2 people, instead of 4.

  Plan to be gone different lengths of time: 2 weeks; 1 month; 6 weeks; 2 months.

  Plan the trip to different parts of the country.

  Arithmetic included: use of the four operations with whole numbers and decimals.

Exhibit Materials

Pupils may make a map of the United States or use a map obtained from some source and on it trace in blue the route going out and in red the route returning.

Pupils may post on the bulletin board pictures showing interesting sights and beautiful scenery which they would expect to see on the trip.

Mileage charts, strip maps, timetables, may be displayed.

SITUATION

Communicating with friends while they are away on a trip. If the man is a business man, he may wish people at home to know where he is each night so that he can be reached if it is necessary.

Questions to Be Decided

How much would it cost to telephone to him each night from Detroit
- (*a*) before 7:00 p.m.?
- (*b*) after 7:00 p.m.?
- (*c*) person to person?
- (*d*) station to station?

How much would different types of telegraphic messages to him each day cost?
- (*a*) 10-word day message.
- (*b*) 50-word day letter.
- (*c*) 25-word night telegram.

How much does an air-mail letter cost? An air-mail special delivery?

Obtaining the Information Needed

Look up long distance telephone rates in front of telephone book.

Call telegraph office for rates for telegrams.

Ask at the post office for the cost of different kinds of stamps.

SITUATION

Jane's family just moved into a new home where Jane can have a room of her own. How much will it cost to furnish Jane's room?

Questions to Be Decided

What is the size of Jane's room?

What pieces of furniture does she want in her room?

What colors does she wish to use in the room?

What will be the cost of furnishings?

Can the furniture be paid for on the installment plan?

How much could be saved if it could be bought for cash?

Are any of the stores advertising sales on furniture?

If so, will it pay to buy at the sale?

Obtaining Data Needed to Answer Questions

Obtain a blue print of the floor plan of a new house for pupils to study and interpret.

Floor plans may also be obtained from the Sunday paper or from several of the well-known magazines concerned with planning, building, and furnishing homes.

Pupils may select from the floor plans studied a bedroom which they may draw to scale, showing the arrangement of the doors and windows.

Pupils may obtain the actual measurements of the space occupied by a bed, a dresser, a chest, and a chair, and draw simple plans of them to scale; they may cut them out to use in deciding upon the best arrangement of the furniture in the room.

Advertisements of bedroom furniture in different woods—maple, mahogany, walnut—may be brought to school and prices may be discussed.

If special discounts are advertised, the amount saved may be computed.

The dimensions of the windows may be obtained from the blueprint, or pupils may measure the length of windows at home, in order to decide how many yards of drapery and curtain material to buy. Prices may be found in the newspaper.

Rug dimensions may be investigated and compared with the size of the room to be furnished. When the proper size is decided upon, prices of different types of rugs may be obtained from the newspaper.

Pictures for the wall may be discussed. Jane may have an unframed picture which she wishes to have framed for her room. The price per foot of frame, the cost of the glass, etc., may be investigated.

Variations of the Situation

The cost of furnishing a boy's room may be investigated. Pupils may imagine that a family moved into a new house and wished to refurnish the living room. The information necessary may be obtained and the total cost of furnishing it with inexpensive, medium-priced, and higher-priced furnishings may be computed.

SITUATION

Bob's father has been making out his income tax report. Bob finds that parents are allowed $400 exemption for each child under 18 years of age.

Bob is 12 years old. He wonders how much it costs his parents a year (*a*) for his clothes, (*b*) for his food, (*c*) for amusements, such as shows, ball games, etc., (*d*) for other expenses, such as dentist, doctor, haircuts.

Questions to Be Decided

Clothing:

What kinds of clothing does he need for the different seasons and for different types of occasions?

What does each article of clothing cost, and approximately how long does each wear?

What does the necessary dry cleaning for a year cost?

Information needed to answer these questions may be obtained from the typical twelve-year-old boy, from his parents, and from clothing-store advertisements.

Food:

What constitutes a balanced menu for breakfast, lunch, and dinner?
What is the average cost of food per day for a family of 4?
What is the average cost of food per day for 1 person?
What is the average cost of food per year for 1 person?

Obtaining Data Needed to Answer Questions

Balanced breakfast, lunch, and dinner menus may be obtained from the household page of the daily newspaper, from recipe books, or from several household magazines. Pupils may obtain some of the necessary information from their mothers.

The recipes for preparing the foods given for each menu are usually given along with the menu.

Current prices of the ingredients needed to prepare each kind of food included in the menu may be obtained from the grocery store advertisements in the newspapers, or from the hand bills distributed from door to door.

Prices of foods not advertised may be obtained by pupils by visiting local grocery stores.

When pupils have computed the cost per person for a typical breakfast, lunch, and dinner, this cost may be compared with costs reported by certain organizations which make a business of ascertaining what a typical balanced meal costs at different seasons of the year.

SITUATION

Jack went with his parents to the Flower Show. They saw so many beautiful garden arrangements and so many lovely flowers, that Jack's father decided to lay out a new garden in their back yard. He asked Jack to help him plan it, buy the materials for it, and plant it.

Information Needed

Approximately how much money can be spent on the garden?
What is the size and shape of the back yard which is to be made into a garden?
What are the soil conditions of the plot to be made into a garden? Is the soil good enough as it is? Does it need fertilizer? Or is it so bad that it needs to be removed and replaced with better soil?
What types of shrubs are to be used in the background—evergreen or flowering shrubs?
How do evergreens and flowering shrubs compare in price?
How far apart do shrubs need to be planted?
How many of each kind will be needed?
What kinds of perennials are available for planting in the spring, and what does each cost?

How far apart should perennials be planted?
How many of each should be ordered?
How much space will remain for annuals?
What annuals should be selected?
Should they be grown from plants or from seed?
What garden tools will Jack and his father need, and approximately how much will they cost?

Obtaining Data Needed to Answer Questions

Ideas for gardens may be obtained from plans given away at the Flower Show, from garden magazines, from the daily paper, and from catalogs from nurseries and seed companies.

From the different garden plans studied, the class may select one, and each pupil may draw it to scale.

To bring in the question of soil conditions, the pupils may imagine that the soil in the proposed garden is the same as in the school yard. A sample of the soil may be sent to the state agricultural department for analysis.

Information regarding the kinds of fertilizer available, the particular uses of each kind, the quantity in which it may be purchased, the amount recommended for each 100 sq. ft. of area, may be obtained from a hardware store or from the garden department of a department store.

When a report is received, the cost of the fertilizer needed to put the soil into condition for a successful garden may be computed.

Descriptions and prices of evergreens, flowering shrubs, and perennials may be obtained from the catalogs from nurseries, or the class may visit a stand or market where shrubs and plants are sold.

If the garden plan selected specifies each type of shrub and plant, these prices may be looked up in the catalog.

Comparative costs of using nursery-grown plants for the annuals, or of planting the seeds, may be obtained by visiting the market or a stand. Plants are sold in large quantities by the "flat" which may contain up to 10 dozen plants, or if purchased in smaller quantities they are sold by the dozen or half dozen. Plants grown from seed are much less expensive—as packages of seeds usually cost from 5¢ to 25¢.

Information about the kinds of garden tools available from which Jack and his father may make their choice may be obtained by a visit to a hardware store in the neighborhood, or from a study of a catalog or advertisements from a hardware store.

Pupil participation in school activities having quantitative aspects. Probably one of the most effective methods of making arithmetic significant for pupils is by capitalizing upon every opportunity which presents itself in the life of the school for the use of arithmetic. As many pupils as possible should be enabled to participate in activities of the following types:

Taking care of sale of milk.
Taking care of sale of tickets for entertainments.
Counting the lunch money to check the cashier.
Collecting bus fare for trip to concert, museum, flower show, etc.
Keeping attendance records and making attendance charts and graphs.
Keeping records of athletic events in the gymnasium and on the playground.
Taking charge of the expenses of a school picnic, or party.
Planning and buying trimmings for the school Christmas tree.
Taking charge of the school bank.
Checking in school supplies and taking care of inventories.
Making out time schedules.
Analyzing school statistics.
Conducting a paper sale to build up the school fund.
Keeping a record of the expenses for the school paper and the receipts from the sale of papers.
Selling Christmas and Easter seals.
Mounting pictures, graphs, and diagrams for the bulletin board—measuring margins carefully.

CONCLUSION

This chapter has set forth the viewpoint that both aspects of arithmetic—that which aims to develop mathematical meanings, and that which aims to develop social significance—may be more effectively taught by the utilization of a variety of supplementary materials, devices, and activities which may or may not be suggested by the textbook or workbook in use. The teacher who accepts this viewpoint will realize that the suggestions given here do not nearly represent all the possibilities that exist for the enrichment and vitalization of the arithmetic course. It is this phase of the work that presents a real challenge to the originality and to the initiative of the teacher.

Chapter IX

WHAT BECOMES OF DRILL?

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At the outset let it be said that the writer of this chapter assumes personal responsibility for the views he expresses. He believes himself to be in substantial agreement with the principles held by the Committee under whose auspices the present Yearbook is issued. He has been at some pains to learn what those principles are and has found himself in sympathy with them. He ventures, therefore, to think that most of the views here presented, even though they may not be found in the same phraseology in the Committee's published opinions, are nevertheless either plainly derivable from them or harmonious with them. Doubtless, however, there are in the following paragraphs some proposals on which the Committee has not yet expressed itself. Perhaps, too, some inferences have been erroneously drawn from positions known to have been taken by the Committee. For all such the writer is himself accountable. To him it would be a source of genuine regret if his personal views, by appearing to have the Committee's sanction, should embarrass a group of thinkers for whom he has the highest respect.

NATURE OF DRILL

Let us start by thinking of drill in its essence without immediate reference to its application. Fundamentally drill is repetition. Its place in learning has been held to rest upon the validity of the Law of Exercise. The current skepticism as to this law has supported a doubt already long held as to the worthiness of drill in a theory of learning. The common-sense experience of the race, however, will be little affected by the proved lack of progress of a blindfolded laboratory subject in a thousand attempts to draw a four-inch line.[1] This may be sheer repetition, but no such sheer repetition has any practical place where teachers and pupils join forces in the interest of learning. In all these practical situations, repetition—not unvarying iteration but action having elements of identity—has always found an honored place and will probably continue to do so. The theoretical question as to the barrenness of mere repetition has, however, important meanings as a guide in education. Some of these meanings we may examine, but we shall not find among them any principle which bids us abandon drill.

We can't do it. Even if mere repetition of an act abstracted in the laboratory from all its connections fails to produce learning, we are still permitted to say that that sort of drill exists nowhere else except in a laboratory and that even there, like a china nest egg, it is as artificial as it is unproductive. If repetition doesn't produce learning effects, then (since learning really does take place) those effects must be produced by something which, though not repetition, is a normal accompaniment of it, such as the sense of familiarity, or the idea of value, or the knowledge of progress, or the stimulus of competition, or the recognition of new uses for old experience, or the onset of deeper insights. In short, it appears that what makes us learn, if it is not repetition, is something inseparably connected with it. Repetition would seem to be a matter of form rather than of substance, something like the schematic arrangement of an outline without reference to the content of the outline, something which is necessary to the result but which isn't active in it—a sort of psychological catalyzer, facilitating the desired reaction while remaining itself inert.

To practical people dealing with classroom matters it may seem unimportant whether children learn by repetition or by something that goes with repetition. The questions at issue, however, are in fact exceedingly important; and they are more important and more challenging than ever before. There has been a shift in emphasis from petty concern with the number of repetitions and the exactness with which one item of drill duplicates another to concern for enhancing the effect of those accompaniments of repetition (largely associated with meaning) which turn out to be the true carriers of learning.

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[1] E. L. Thorndike, *Human Learning*, pp. 8-10. D. Appleton-Century Company, New York. 1931.

For example, we no longer anxiously speculate as to the number of repetitions of $5+7=12$ that are necessary for mastery. If we do not know by personal trial that any such quest is futile, the evidence concerning the Law of Exercise convinces us that it is not the *fact* but the *manner* of repetition that counts. We are therefore sure that one occurrence of $5+7=12$ does not equal in learning effect another occurrence of it and that in the absence of meaning some of these occurrences may be useless or even inhibitory.

Again, with better evidence concerning human learning, we are not so painfully concerned that each element in a series shall be of exactly the same kind as every other. In particular we no longer analyze processes into such an appalling number of categories—the division of integers into 34 "unit skills" and the adding of fractions into 42. The object was to provide under each of these numerous classifications a series of items of identical type for use in drilling the pupil on that type. The items, being so closely alike, were admirably repetitive but were deficient in the learning overtones which we now regard as indispensable. Moreover, it was neither necessary nor expected that the pupil would understand what he was doing. It was the drill, the repetition, that counted.

Change in concept of drill. To describe these matters is to realize that a real change has taken place in our theory of learning—a change which is already beginning profoundly to affect practice. In respect to drill, the truly important problem for us is not to banish it but to give it a new meaning by loading it with the elements which make it effective. Those who have been disappointed with drill and the pitiful results of it are foolishly saying "let us have none of it," just as those who have (for curiously identical reasons) been disappointed with primary grade arithmetic have demanded that it be thrown out. This is too simple a solution. Progress does not take place in that manner. Old ways have lessons for us; and this way certainly has. Drill should not be dropped or decried. To do so would be to handicap the schools with a new fad.

Rather let us, first, seek new ways of making drill do a better job. In the light of our present knowledge, we may be assured that the more nearly drill approaches sheer repetition the more barren it will be, while, on the other hand, the more it involves the accompanying conditions already mentioned the more fruitful it will be. If, therefore, we ask ourselves what place drill may properly have in a theory of learning, we shall find that the answer, though much more difficult than used to be supposed, is nevertheless more vital and satisfactory. A study of this aspect of drill will lead us into a territory rather different from that which has hitherto been explored.

Secondly, let us apply the concept of drill to kinds of learning with which it has not usually been identified. It is now several years since the basis of the usual conception of drill was knocked from under it. It is time for the necessary conclusions to be drawn. The deprecatory way in which many of us think about drill would disappear if we gave it a truer and more liberal meaning. And this we certainly have a right to do now that we put our faith more in the meaningful accompaniment of drill than in the repetition *as such* which was so long held to be its productive characteristic.

Drill in thinking. Success in thinking is facilitated by taking thought on numerous occasions, i.e., by drill in thinking.[2] The mastery of a generalization is won by frequent experience to which the generalization applies, and frequency of experience is a form of drill. The love of learning, the scientific attitude, and the ability to get on with people are products of learning in which drill plays a part. Perhaps "drill" with its bleak connotation should be replaced by another word, perhaps by "practice" or by "recurrent experience." Dewey's concept of growth through reconstruction of experience is impossible without the idea of recurrence. The things we most value in life are maintained by action—and best maintained by repeated action. Lurking in the distinction that we too often make between drill on the one hand and ideational learning on the other, there is something of the dualism in philosophy against which Dewey has fought so valiantly. In his *Quest for Certainty* he argues against the age-old separation between theory and practice. In doing so he says, "We should regard practice as the only means (other than accident) by which whatever is judged to be honorable, admirable, approvable can be kept in concrete experienceable existence." Of course, Dewey's term "practice" is not quite our term "practice" and the quotation may seem on that account less appropriate than it really is. For what is the practice that we oppose to theory? Is it not action evolving out of action, and forming itself into patterns, all with highly repetitive parts and processes? It is difficult to maintain any other position.

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[2] The reader should note carefully that beginning at this point the word "drill" is given an unusually extended meaning. In its traditional sense, which is to say, in the sense of practice with as little variation as possible, "drill" could not properly be used in these sentences. The essence of the word as it is employed in this chapter is that not one experience, but several different experiences are requisite to thorough learning, whether the thing learned be the richness of an important understanding or efficient use of a more or less mechanical skill. See the writer's clear statement of the meaning of drill on pages 196-197. (*Editorial Board*)

Two preliminary points regarding drill have now been made: first, that the efficacy of drill lies in something other than its repetitive nature; and, secondly, that we are therefore justified in applying the concept of drill (as recurrent experience) far more widely than has hitherto been the case. Our position is not that drill should be avoided but rather that it should be made more intelligent in the fields to which it is now applied and that it should be applied still more widely. Indeed it is doubtful if those who vigorously denounce drill are wholly sincere. It is possible to take the view that this present widespread deprecation of drill is a sort of psychological revenge whereby the case is overstated, not calmly nor entirely in the interest of truth, but rather with the purpose of utterly extirpating a hated doctrine.

Application of principles to arithmetic. Let us now apply our two principles to the field of arithmetic. The first principle will bid us seek out the active, energizing, vital aspects of drill—the concomitants of repetition which make the repetition effective. The second principle will carry us into fields of arithmetic or, in some cases, into *kinds* of arithmetic which are not ordinarily associated with drill techniques.

For the sake of brevity it may be well to eliminate from this chapter much that might be said about drill, first, because it has already been said a hundred times, and, secondly, because we shall here be concerned with many other things. Accordingly this chapter will avoid discussing the following points copied from Burton who quoted them from Parker who got them from . . .?

(1) A correct start followed by correct practice must be ensured. Speed should be subordinated to accuracy.
(2) Zeal, interest, and concentration of attention must be secured and maintained.
(3) Feelings of satisfaction and dissatisfaction must be considered, as they vitally condition the results of a drill lesson.
(4) Avoid wastes of time on accessory and nonessential processes. Drill must be on the association or skill involved.
(5) The facts drilled on in games and devices must be applied in real situations.
(6) The drill periods should be short and distributed over a considerable length of time.
(7) Learn under some pressure.
(8) Use ready-made drill systems.
(9) In memorization there should be an analysis of the thought content first. Correct recall should be the principal method used; the *whole* instead of the *part* method.

SOME VITAL ASPECTS OF DRILL

Reference has already been made to the ineffectiveness of sheer repetition. The evidence on this point is conclusive and its acceptance, especially on the part of those who adopt an organismic psychology, has been all but universal. Thorndike himself has furnished impressive experimental proof. On the other hand, Koffka, Köhler, Hartman, Perkins, and Wheeler—the whole Gestalt group—seem to unite in emphasis upon the barrenness of mere iteration. They note the deadening effect of bald repetition when they attribute plateaus and recessions in the learning curve to what they call "irradiation"—an influence which exerts a sort of paralysis upon the learner. It is not difficult for anyone who has taught arithmetic to recognize the onset at various times of this so-called irradiation. Children who knew (or at least could *say*) $5+7=12$ and $8+6=14$ yesterday do not know these facts today. Those who were apparently coming along nicely with their long division go all to pieces. One cause of this is sheer unvarying drill, and too often the remedy applied is more drill of the same kind.

The bad effects of unvarying drill, however, are by no means confined to immediately observable errors in the process which is being repeated. If our drill is devoid of intellectual content, if it is merely repetitive, it is an encouragement to divided attention or double-mindedness. Thus we may find ourselves drilling on nothing so effectively as on the double standard, the loss of energy, and the habit of self-deception which Dewey brings forcibly to our attention in *Democracy and Education* (page 209). He speaks of the seriousness of "exaggerated emphasis upon drill exercises designed to produce skill in action, independent of any engagement of thought—exercises having no purpose but the production of automatic skill." Continuing, he says, "Nature abhors a mental vacuum. What do teachers imagine is happening to thought and imagination when the latter get no outlet in the things of immediate activity? . . . They follow their own chaotic and undisciplined course. What is native, spontaneous, and vital reaction goes unused and untested, and the habits formed are such that these qualities become less and less available for public and avowed ends."

Since this passage was written, some twenty-five years ago, thousands of educators have no doubt read it. What do they make of it? Very little it seems. A reasonable search conducted in the interest of this chapter in the literature of drill and allied subjects has failed to reveal any clearly marked effect of Dewey's grave warning. Yet the effect lies all about us in the product of the teaching of arithmetic in the schools.

Observe the child when he enters the first grade. He knows more than many of us think he does about number. Moreover his knowledge is remarkably functional. He can use it; he can apply it in concrete situations. For example, give him this problem: "If you had five cents and Father gave you three cents, how many cents would you have then?" This situation is fully understood by him and he can react intelligently to it. He will probably reply, "Eight cents." On the other hand, if you ask him, "How many are five and three?" he will be likely to disappoint you.

Now consider the same child, or another one like him, after he has had a few years of schooling. What does his teacher say of him now? The following is taken from a number of questions asked by teachers of arithmetic in a certain city: "My children can do abstract work but they cannot solve problems; what can I do?" Those who have taught arithmetic recognize in this question a typical difficulty. Computational skill, no matter how highly developed, is unsatisfactory. In described situations involving number relations the pupil is at sea in spite of his skill in computation.

You will recognize the contrast between the child entering school and the child who has received the benefit of schooling. You observe the complete reversal: on the one hand, the ability to sense the described situation and to act intelligently in reference to it; on the other hand, nothing but a useless skill—useless because in life no one computes except for the purpose of solving a problem. It appears therefore that we have done something to the child, and that what we have done is not good. Our drill has been without intellectual content and has produced in too many instances a purely formal ability.

The permanent effect of the school's devotion to a barren type of drill—to a drill limited almost exclusively to abstract computation—is evident in the arithmetical illiteracy of our people. Everywhere we meet adults of otherwise reasonable training who are almost helpless in the presence of numbers. They are unwilling to read books and articles which present quantitative ideas. They will not listen to an address which makes similar demands upon their thinking. They blandly tell us that they are "no good at figures," in spite of the fact that persistent drill in figures formed much of their schooling. As we know, however, it was the kind of drill which Dewey again vigorously denounces: "a drill which hardly touches *mind* at all—or touches it for the worse—since it is wholly taken up with training skill in external execution." In this connection Dewey does not fail to invoke one of his favorite principles when he continues: "Practical skill, modes of effective technique, can be *intelligently*, non-mechanically *used* only when intelligence has played a part in their acquisition."[3]

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[3] John Dewey, *How We Think*, p. 63. D. C. Heath Co., Boston. 1933.

One is reminded in this connection of the doctrine of "traces" in organismic psychology. Experience produces traces, and one of their characteristics is *availability for new purposes.* How the traces thus become available is still obscure. "However," says Koffka, "one conclusion seems fairly safe: conditions which make a trace more and more available for mere repetition of one process will often make it at the same time less available for other processes. Thus the educator should be very conscious of his aims when he decides whether to apply drill or not. Drill will no doubt make the traces more and more available for one kind of activity, but it may at the same time narrow down the *range* of availability."[4]

From this we readily conclude that "range of availability" is of the utmost importance in connection with the activity we call drill. Actually, however, if we may judge from published drill materials and from observation in the classroom, this characteristic of desirable drill is entirely disregarded. The crux of the question is the old and ever-true idea of *function,* of the use you make of a thing. Let it be understood, however, that in deprecating a certain prevalent kind of drill we must not fall into the error of trying to get rid of drill. We must rather improve it and make it useful.

It is important to be sure, first, *when* (that is, at what times and under what circumstances) we want to use drill and, secondly, to what fields we wish to apply it. Two aspects of the first part of our problem seem to present themselves, namely, readiness on the part of the pupil, and the organization of the drill material itself.

READINESS FOR DRILL

The time has probably gone by when anyone would support a drill theory which proceeded on the assumption that the child can profitably be subjected to drill without being prepared for it. Here, as elsewhere, his readiness, his receptivity and sensitivity to what is about to take place, must be provided for.

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[4] K. Koffka, *Principles of Gestalt Psychology*, p. 547. Harcourt, Brace and Company, New York. 1935. Koffka is here using the term "drill" in its narrower sense, as a procedure applicable to mechanical matters. He does not touch the question of whether repetition of experience may not be actually planned to give a wider "range of availability."

In the first place, the dynamic force of *purpose* should be invoked. Drill without purpose is likely to be empty. As everybody knows, the presence of purpose insures an effective organization of action. All the parts of the action take on unity and meaning through their relevance to a larger objective.

How shall the appeal of purpose be secured in the administration of drill? By many means, no doubt; but mainly by making the practice a part of something the pupil recognizes as useful. Most potent will be the problem which arises naturally in the child's life. Would that there were more of these and that we had better means of capturing them. They are not, however, available in sufficient quantity and variety to serve our need. We shall therefore be obliged to fall back upon the described situation; that is, upon the well-known verbal problem. At its best this is not to be despised. Indeed, under present school conditions, it is more than likely that the body of verbal problems offered in an arithmetic course can become the most significant part of it. By no means all of these problems will involve computation. All of them, however, should relate directly or indirectly to the pupil's interests. Their solution should appeal to him as worth while. The abstract drill needed to insure solution then becomes significant. This drill, however, may involve many steps. In order that these steps may be seen to have bearing upon the larger purpose, the drill should not only be preceded by problems but also be paralleled by them.

Larger purposes than those supported by verbal problems must, of course, be entertained; but we shall drop the matter here lest we develop an *excursus* on this attractive subject. We shall content ourselves by repeating that in the matter of readiness for drill the child's sense of purpose takes first place.

In the second place, readiness for drill is enhanced if the child has a sense of the *value* of what he does. This is closely related to purpose. Indeed if we accept "fitness for a purpose" as a definition of value, the child's awareness of the relation of his task to his purpose *is* his sense of value. As a matter of theory, however, it is worth mentioning that whereas personal aims may have no ethical reference, the value of an action cannot escape such reference. Thus a pickpocket has base purposes and makes efforts (including drill) to attain them. But his practice cannot in any general sense be said to have value. Purpose then is personal; value may not be. Accordingly when we say that the child should have a sense of the value of any drill which he is required to undertake, we provide for something beyond his self-regarding purposes. At any rate, if the pupil has not reached the point where he sees value in a particular type of drill, then he is to that extent unready for it.

In the third place, readiness depends upon the *confidence* of the pupil in his ability to do the task in question. He should have the habit and expectation of success—a condition brought about by a sustained policy of always preparing him adequately for the drill he is to undertake.

In the fourth place, if the drill is not so linked to previous experience as to arouse a feeling of *familiarity,* then the child is not yet fully ready for the drill in that form. This principle is evidently useful in organizing drill, but it is also a principle of readiness. Not only should we arrange drill units so that familiar elements appear as linkages, but we should also endeavor to prepare the pupil for the recognition, emotionally as well as intellectually, of these linkages. Our drill should be progressive in range and difficulty, but it should never be really new. If it is, then (according to our point of view) either it is bad drill or the child is not ready for it.

In the fifth place, if the pupil is not prepared to some extent to take charge of his own drill—the material, if necessary, being provided for him—he is not yet fully ready for it. His unreadiness may be due to lack of experience in taking responsibility or it may be due to a lack of understanding of the field covered by the drill in question. In either case, since the objectives of drill are personal, and since therefore they will be best assured when the person most concerned takes an active part in attaining them, readiness implies the ability to see the need and to help in applying the remedy. The best drill has large elements of *self-direction.*

With all the skill at our disposal in organizing drill, there will be times in actual practice when the ideals and motives above referred to will be insufficient to carry the work forward and when effort must be brought to bear. In the sixth place, therefore, readiness implies that the pupil have at his command habits of *attention* adequate to the requirements. At the same time, it should be noted that attention, though an item in readiness, is also, like several items already mentioned, an item to be considered in organizing a drill procedure. It is for the purpose, among other things, of reducing the demands upon attention that short practice periods, time limits, distributed practice, and perhaps learning by wholes, are favored. But when all proper arrangements of this sort are made, a certain span of attention is still required. The ability to bestow, under the conditions present, the degree of attention required is obviously an item in the pupil's readiness.

The things we offer our pupils for practice cannot, therefore, be considered apart from the readiness of the learner for them. We have noted six aspects of readiness: a purpose in harmony with the material, a sense of its value, self-confidence in undertaking it, a feeling of familiarity with it, ability to take some personal responsibility in reference to it, and habits of attention adequate to meet its demands. Each of these, though viewed here as a matter personal to the child, has perhaps equal bearing upon the character of practice and its organization. Thus drill material should have purpose and value for the learner; it should inspire confidence and a sense of familiarity; it should provide in some degree for self-direction; and it should be reasonable in its demands upon attention.

THE ADVANTAGES OF ORGANIZED DRILL

To the writer, the advantages of an orderly arrangement of learning activities are unquestionable. He cannot subscribe to the doctrine that makes education a succession of improvisations. To him "the basic material of study cannot be picked up in a cursory manner." It is true that occasions which cannot be foreseen arise, and always will arise as long as men are free to think and act; and it is equally true that these occasions should be utilized. No theory of education which recognizes its experiential character will deny the vigor and propulsion of such occasions. The point, however, is that the dynamics of the unexpected incident should be used in the service of developing a continuing line of activity. This is far different from trusting to a series of fortuitous occasions to provide the material of learning. The point of view of the writer, therefore, is that which Dewey expresses in his *Experience and Education* (see especially pages 95-103). According to that view, the truly underlying ideal in education—whether we are talking about drill or any other aspect of education—is not merely, nor even predominantly, a series of discrete contacts with reality, however vivid, but rather the progressive organization of knowledge. Dewey is no doubt thinking of many a mistaken follower of his when he remarks: "In practice, if not in so many words, it is often held that since traditional education rested upon a conception of organization of knowledge that was almost completely contemptuous of living, present experience, therefore education based upon living experience should be contemptuous of the organization of facts and ideas."

When experience is repeated we have drill in the conception of the term employed in this chapter. If organized materials—not necessarily as the starting point but as an objective—are desirable, then organized drill is desirable. The very doctrine of the continuity of experience as a moving force implies not only development, not only novelty, but also repetition.

We must use the occasions of the living present; we must recognize the ongoing character of experience; we must invoke repetition as well as insight; we must look toward a reorganization—which, in spite of the prefix, does mean organization—of experience; and we must seek, if men are to rise above the fortuitous and the occasional, the orderly arrangement of knowledge.

These things simply cannot be had without exercise, without practice, without drill; and every argument for the organization of educational efforts toward the accomplishment of purposes bids us likewise organize our drill for the same ends. This demand will play down rather than play up the virtues of improvisation and will declare with emphasis and clarity that improvisation is at once the *bête noire* of progressivism and the resort of the lazy teacher.

THE INCIDENCE AND ORGANIZATION OF DRILL

The question of when to use drill involves, among other things, the question of its incidence. One general rule can be laid down. It is that drill should follow a certain degree of understanding on the part of the pupil and should be designed to increase that understanding as well as to impart such qualities as ease, fluency, speed, and skill.

The more nearly formal the drill, the more complete the understanding of the pupil should be before he is subjected to it. It is too often assumed that subject matter is understood when it has merely been memorized in a formal way and can be reproduced. A better doctrine will assert that nothing is really known unless it is understood. If, then, understanding is essential, the less we are able to provide for it *during* drill, the more carefully we must provide for it *before* drill.

Two procedures, therefore, are open to us and both of them must be employed, although the emphasis on the one or the other will differ according to circumstances. The first procedure is to develop carefully and concretely the meaning of the material which is going to be used in a repetitive way. The second procedure is to provide meaning during the course of the practice itself—a procedure the details of which need not delay us at this point. The character of these two procedures—both being in the interest of understanding—may be illustrated by reference to the number combinations. Our objective when we include these combinations in our course of study is their automatic mastery. Our approach, however, to practice in the combinations may well involve a far longer and more inventive treatment of them, using concrete detail and manipulation with much thought as to the nature of the processes involved, than has hitherto been expected.

The practice itself, should be less abstract and more clearly in the service of meaning.[5] Relationships among the items of drill should be eagerly exploited. Perhaps even more important is the utilization of meaningful material to supplement the abstract material. It is entirely likely that the solution of meaningful problems is itself a powerful type of drill, even with reference to number combinations and processes, to say nothing of the benefits of such drill in increased ability to solve problems. We are not without evidence that one of the best ways to develop among pupils the ability to solve problems is to practice problem solving.

The configurationists are especially strong in their emphasis upon insight. Summing up this matter, Hartmann goes so far as to say: "Insight, then, takes the place of practice or repetition as the key word in a configurationist picture of learning."[6] This does not deny the value of practice. It merely puts insight in the more important position. Incidentally it would seem that this question of priority is rather futile, like the fabled quarrel between the mouth and stomach as to which is the more important organ of the body. The frequency with which acts are performed unquestionably affects progress in any performance. At the same time, the protest of the configurationists, while it may by its vigor fail to present a balanced judgment, is nevertheless wholesome. Something like it had to come. The thousands of teachers and millions of pupils who have been disappointed by misplaced confidence in drill and especially by a use of drill at the wrong time are entitled to a vigorous corrective. The insight or, in Gestalt language, the configuration is the important thing. "Repetitions without the achievement of a configuration," says Koffka, "remain ineffective whenever they are not positively harmful."[7]

In arithmetic the extent to which the school went in robbing its offerings of meaning—even claiming this as a virtue—can be inferred both from the literature of ten or fifteen years ago (to which, alas! the writer contributed) and from observation in many a classroom today. Repetition, in theory and in practice, has been held to be the "key word" in learning. As a consequence, meaningful relationships within the material to be learned have been considered not only useless but probably harmful (through "interference"). For example, in arranging the number combinations for learning purposes, it has been regarded as good practice to keep apart those in which the same number appeared. According to this doctrine, $4+5=9$ could be followed by $8+3=11$, or by $6+6=12$, but not by any combination containing 4, 5, or 9, because (to be specific) it was held that $4+5$ might later turn out to be 7 instead of 9 if it were immediately followed by $4+3=7$. With the present organization by relationships, common elements are sought rather than avoided, and a number of helpful groupings of number facts are suggested.

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[5] Emphasis upon relationships is scarcely characteristic of the traditional meaning of drill. In his advocacy of more (and better) drill the writer leaves small comfort for those who have relied upon sheer repetitive practice to accomplish ends which as the writer elsewhere shows (e.g., page 197) cannot possibly be so achieved. (*Editorial Board*)

[6] George W. Hartmann, *Gestalt Psychology*, p. 171. The Ronald Press Co., New York. 1935.

[7] K. Koffka, *Growth of the Mind*, p. 234. Harcourt, Brace and Company, New York. 1925.

Drill on wholes. In the interest of putting meaning into our drill procedure, let us deal with wholes as far as the child's experience permits rather than with minute parts. The extreme subdivision of topics is contrary to every modern conception of learning. Indeed, it would seem highly desirable that, for example, each operation in arithmetic should be begun and for some time practiced as a manipulative and as a thinking process. Each operation (division, for example) would then come into view *as a whole* and in its essential operational meaning. Thus approached, it would be far better understood, with the result that its application in concrete situations and in the semi-concrete situations of verbal problems would be more successful. Then the fifth grade teacher would not be so likely to find that her pupils could "do abstract examples but not concrete problems." Each abstract process, as a process corresponding to a way of thinking, would have its own inescapable meaning.

This is not the same as the proposal, sometimes made, that all processes and procedures should be explained. There are many ways of using numbers which some children cannot understand (at least at their present level of maturity) no matter how well they are explained. These ways sometimes depend upon the number system or upon ideas of a generality quite too advanced. Nevertheless the processes themselves are needed. Moreover, the meaning of these processes as such can be understood as ways of thinking about numbers and can be practiced as ways of handling counted or measured things long before the manipulation of digits (as in carrying or borrowing or finding the common denominator or inverting the divisor) can be understood at all.

Overorganization. Are we not in some respects going too far in organizing our drill materials? Do we not spend too much time getting ready to do something important without actually doing it? To many thinkers it seems that the school, long under the dominance of atomistic ideas, has developed a passion for petty detail. There are those who say that the school gives most of its attention to the background of thought while "thinking practice is always just around the corner."[8] Backgrounds of thought are needed; but a school geared to the "stuffing idea" will seldom get past them. A little less meticulousness in our drill organization and a little more largeness of view will be helpful.

For example, we do not need to lay so much groundwork for remote and contingent uses. The next move in the arithmetic curriculum—next, we mean, after the current tendency to transfer the completion of certain topics to higher grades—may very well be the ousting of some of the present "business practice" from the seventh and eighth grades. This present trend to defer the completion of topics hitherto taught in the lower grades is already congesting the upper grades, and we are soon going to ask ourselves whether we want to teach taxation, banking, and investment to boys and girls of fourteen and fifteen. It happened (by an odd coincidence) that the reductionists who worked over our arithmetic course in the "teens" and twenties of this century were themselves especially interested in business as a source of subject matter. They therefore held it to be a sufficient reason for retaining a given topic that it was good business practice. Many years must elapse between the study of it and any possible use of it; but it was "business usage" and that was, and has long remained, a sufficient justification. This maladjustment can easily be taken care of in the high school course in arithmetic which is now developing. Meanwhile it is pertinent to raise the question of whether we do not spend too much effort getting ready to teach and then actually teaching things which the child will not have use for until after he has forgotten them.

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[8] Robert W. Frederick *et al.*, *Directing Learning*, p. 481. D. Appleton-Century Company, New York. 1938.

The idea of thoroughness. The limitation of practice to symbols and tools and the consequent neglect of practice in higher processes is closely related to the conception of thoroughness which the school has long entertained. According to that conception, a teacher cannot conscientiously allow a pupil to move from one topic to the next until the first topic has been completely mastered. There is good reason to suppose that this idea is responsible for much loss of time and morale. Yet anyone rash enough to hold a contrary view risks the scorn of the doughty advocates of thoroughness.

Do you remember the accuracy standard proposed some years ago by Thorndike for the number combinations? He said: "An ability of 199 out of 200, or 995 out of 1000, seems likely to save much more time than would be taken to acquire it, and a reasonable defense could be made for requiring 996 or 997 out of 1000."[9] The more severe standard, namely 996 or 997 per 1000, was quoted with approval by Brown and Coffman, coupled with the admonition that only when responses to the combinations possess a high degree of accuracy will the pupil be "ready to take up successfully the more difficult operations."[10] There is a whole philosophy of education in this idea. It is the philosophy of 100 per cent accuracy in the "fundamentals" of arithmetic. It is the philosophy which forgives an error of statement but not a misspelled word in a letter. It is a philosophy which would keep the school grinding away at petty tasks without coming to grips with things of greater importance. And it is a philosophy which is passing and deserves to pass.

In arithmetic, with particular reference to the idea of "a high degree of accuracy" at each step before the next step is taken, it is clear that this common concept of thoroughness quite disregards the value of drill in application. To a greater degree than is allowed for in this type of thinking, 7 + 5 and 16 − 9 and 8 × 4 and 42 ÷ 7 are learned in use—for example, the addition facts in column addition and in carrying in multiplying, the subtraction, multiplication, and division facts in long division, and all the facts and processes in problems.

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[9] E. L. Thorndike, *The Psychology of Arithmetic*, p. 108. The Macmillan Company, New York. 1922.

[10] Joseph C. Brown and Lotus D. Coffman, *The Teaching of Arithmetic*, p. 77. Row, Peterson and Company, Evanston, Ill. 1924.

Good drill involves change. All drill if it is effective must, to a greater or less degree, be progressive. It must tolerate change and in many kinds of drill consciously provide for it. Some of the change is, of course, in the learner. He progresses, increasing his skill and his awareness of his skill. When he ceases to improve he is likely to lose interest and to remain stationary for a considerable period, thus entering upon a plateau during which he frequently quits altogether. We say he has lost morale.

Another type of change is in the goal to which the drill is directed. Normally, at least as long as progress is made, the goal entertained by the learner is growing and developing. It becomes more advanced, more mature, more difficult—more, more, more, including more firmly entertained. Drill, then, is not well organized unless it is motivated by increasingly better and more fully accepted goals. This is one antidote to the monotony of unvarying repetition. In some cases it is enough to have *repeated effort* to attain or move toward a goal, allowing the content of the effort to change. In such case it may be the change (which, however, could not have taken place without the repetition of effort) that accounts for most of the learning.

A third type of change is in the events themselves. They should not be all alike. Theoretically, of course, they never are in all their details and circumstances, but they can be sufficiently alike to deaden the learning. This matter of providing in our drill for steady progress is a major problem in learning. It may be met by changes in the learner or by changes in the goal as already suggested. But a change in the material itself is desirable. Moreover, when made it reacts on the other two—the morale of the learner and the goals he entertains.

Change in the material, however, is not enough. The change should be progressive; it should have direction. Such a type of change has been called "pacing." It calls for presentation to the learner of tasks which increase with his ability. The increase will usually be in point of difficulty, but it may also be an increase in range, variety, or importance.

We have already noted in this discussion that in its incidence drill in arithmetic must follow understanding; that an appeal to understanding should also be made throughout its course; that whole ideas rather than parts of ideas and petty details should be practiced; that in this latter sense our drill is often overorganized in the interest of a rigid and too literal idea of thoroughness; and that, finally, drill must be progressive as to (*a*) the learner, (*b*) the goal, and (*c*) the task.

External organization. A higher type of organization concerns the external articulation of one drill unit with another. Here we find reasons for the disrepute into which drill has fallen. The external organization of learning units has been based upon the plan known as the "drill theory" whereby parts are divided into parts and these in turn into still smaller parts and so on until "unit skills" or some irreducible item is supposed to be reached. These items so highly fragmented can have little meaning because each is so far removed from the whole which alone can give it meaning. The pupil is to be drilled on these items separately and then is to build up for himself the larger parts of wholes. This scheme is discredited by most of those who have recently expressed themselves on the subject. It is apparently the least promising type of organizing a series of drills.

The reverse process of beginning with the largest possible wholes—these wholes, of course, being necessarily rudimentary—has already been suggested. Another process which bids fair to give good results is that which resorts to the purpose of the learner and his goal-seeking activity rather than to the logic of subject matter as a principle of organization.

A QUESTION OF TERMINOLOGY

In the foregoing pages of the discussion an attempt has been made to look more closely than is customary into the nature of the repetitive procedure which, at least when narrowly applied, has been called drill. It has been shown that this repetitive procedure, if it is to serve well even its conventional purposes, should not be the mechanical grind that many have made it. In our few remaining pages we shall show that when regarded in this more humanistic way, the procedure at once becomes serviceable for purposes not usually associated with it.

Meanwhile what term shall we employ to cover this wider field? Shall we narrowly restrict the term drill while thus extending its basic idea? If so, what other name shall we give to the extension? Shall we call it *practice?* This term would serve well enough if it were not already used to mean the same as drill in the narrower sense. We constantly use the expressions *practice material* and *drill material* to mean the same body of exercises. Concerning a textbook or a workbook, we ask how much practice or how much drill it offers on long division; and in either case we mean the same thing. If, therefore, we decide to call the wider uses of a repetitive procedure *practice* and the narrower uses *drill,* we make a distinction between terms which have hitherto covered the same ground. In answer to "What becomes of drill?" it is hardly satisfactory to say, "It becomes practice."

Shall we apply a wholly different term to the broader idea which we have in mind? The writer admits his unwillingness to do so. Educational terminology is already too elaborate; and those who are engaged in extending it are of doubtful service to the cause they represent. Their thoughts would be better understood if put in plainer words.

It is therefore proposed that we keep the word drill, allowing it to take on wider meaning as the objectives of arithmetic themselves widen. The writer sees, for example, no disadvantage and some advantage in the expression "drill in thinking." Moreover, he suggests that it is a commentary on our exclusive concern with intelligence if we cannot accept "drill in tolerance of—" or "drill in appreciation of—." These things are as surely products of learning as knowledge of the interest formula or the ability to write a check. They are likewise improvable, as other learnings are, by recurrence of appropriate experiences, that is, by drill.

THE FIELD OF DRILL

The reduction in the arithmetic course during the last thirty years has been mainly a reduction in computation. In grades one and two the entire subject has either been omitted or devoted to the development of concepts and number relations by concrete methods. In all the grades some topics (mostly computational) have been dropped and others have been postponed. Still other topics—as common fractions, measurement (denominate numbers), insurance, taxation—although retained, have been changed so as to represent ideas rather than mere figuring.

What is taking the place of computation? What should take its place?

For one thing, we are replacing computation from arbitrarily given data with activities in which pupils obtain the data as well as the answers. Instead of giving the shadow length of a tree and the height and shadow length of a post, in order to find the height of the tree, we send a group out to make the measurements and then do the computing. The latter becomes an interesting climax; but, as far as time and effort are concerned, it is distinctly subordinate. Of course, on this expedition pupils will make repeated measurements and computations of shadows and heights. Even the expedition will be repeated. Thus drill will be provided for—drill in the whole activity, not just in the figuring. A committee may make a crude transit out of a camera tripod, a protractor, and a clock hand. Then data may be gathered for still further indirect measurements. (N.B. Similar triangles will be better understood.)

Other ways in which, by getting the data as well as the answer, a superior type of drill may be applied are: finding circle graphs in newspapers or magazines and interpreting them; making out applications for money orders (or, better still, sending the money); finding distances in miles on a map and using the distances for some purpose; getting averages by first making the needed measurements; opening a postal savings account; working out problems after making weather observations; finding areas of figures drawn to scale; finding the value of π by measuring round objects such as a wastebasket, tin can, lamp shade, water glass, etc.; reading gas, electric, and water meters to find the amount of the bill; window shopping (or newspaper reading) to collect discount data for use in problems. If, with the progressive variation already referred to, these and similar activities are done several times, in other words, if they become the subject of drill, the learning effect will be greater than a single experience will permit. Moreover, with the reduction in computation which characterizes the present arithmetic course, there is room for this sort of thing though perhaps the full fruitage of such work cannot be had until the course becomes more definitely a secondary school subject than it now is.

QUANTITATIVE THINKING

Another type of work which is taking the place of computation in arithmetic, and to which drill ought to be applied, is quantitative thinking, that is, thinking in which the data for thought are numerical. One type of thinking (not necessarily quantitative) is *comparing*, another is *arranging*, a third is *classifying*. We may compare the French and British governments, or Washington and Lincoln, or liberty and equality. We may arrange events in the order of their occurrence. We may classify vegetable foods as roots, flowers, bark, fruit, seeds, etc. But if we compare, arrange, or classify on a numerical basis such as weight or money value, the thinking which our action involves or sets going is quantitative thinking.

One is tempted at this point to say more than should be said about comparison as a type of quantitative thinking. Much important arithmetic gathers about it. First, we have comparison by subtraction—the idea of more and, contrariwise, of less. Secondly, we have comparison by division or ratio—the times and the part idea. And thirdly, we have a combination of these two in virtue of which we find the difference between two magnitudes and express the ratio of the difference to one or the other original magnitude, e.g., from the facts that the population of Seattle in 1910 was 237,000 and in 1930 was 366,000, we may say that the increase in population was 54 per cent. At any rate, a great deal of the practical arithmetic of life comes under the heading of comparison. Various ways of expressing comparison are constantly found in periodicals. In order that pupils may understand them, drill in this form of thought should be provided.

As to arrangement, suppose we have a table of measurements just as they were made. In order to understand them, we must first bring them into some sort of order. The obvious one is arrangement from smallest to largest (or vice versa) according to size. This is, perhaps, not so much a matter of thought as it is a basis of thought. From this arrangement we may select (selection is another thought type) the middle measure as a very good expression of the general weight or central tendency of the series.

Again, arrangement of data often leads to classification, the third of the thought types above mentioned. Measures may be grouped as more or less than a critical measure. They may be put into a series of consecutive groups, as those between 20 and 29, those between 30 and 39, etc. Sometimes classifications become stereotyped and thus provide a ready-made instrument which then becomes widely used and on the basis of which valid comparisons can be made, for example, the census bureau's classification of cities according to population.

The types of thinking, quantitative or otherwise, are numerous, and during the arithmetic course drill should be had on all of them. In addition to the three already mentioned, the following list may be suggestive: giving examples or instances like, emphasizing, judging or evaluating, adapting (to a related purpose), discriminating, approximating, inferring, defining, planning, choosing, summarizing, reproducing (acting under the guidance of a number idea), identifying (recognizing and naming the number idea when its objective conditions are presented), generalizing, applying, analyzing, choosing, explaining, illustrating, synthesizing. Of course the four processes of arithmetic, as well as proportion, are likewise thought types. Children are drilled enough on them in all conscience (except on proportion), but the drill is not on the thought which they represent but on a particular arrangement of the digits of the Hindu-Arabic system which happened to prevail after the sixteenth century and which gets the answer.

ATTITUDE TOWARD NUMBER

As long as the greater part of the time and effort of the pupil is devoted to the trivialities of arithmetic, a large and generous attitude toward the subject cannot be expected. If his days are spent in a purposeless round of drill in computation, if the chief motive offered him is the far-off and uncertain prospect of going into business, of buying insurance, of paying taxes, and of investing money, he will have as little to do with arithmetic as he can. It will seem to him an especially dreary subject and his attitude toward it, not only in school but afterward, will be one of aversion.

Attitude, however, is a by-product. One does not, except in a bad sense, practice an attitude. Yet attitudes are acquired or learned; we are not born with them. They are, in Kilpatrick's phrase, "concomitant learnings." This means, not that we practice or drill on desirable attitudes directly, but rather that our drill, though otherwise directed, must bring repeatedly into play these desirable attitudes.

This is part of the reason why drill should as often as possible be general rather than specific, and thoughtful rather than mechanistic. For example, children are drilled on interest problems using the formula $i=prt$, but few of them either see or use the indirect cases or the corresponding formulae for p, r, and t. Are we told that *in business,* principal, rate, and time are seldom computed? True enough, but there is a good answer to all that if there were time to give it.

Children work upon specific numerical ratios, yet they fall far short of sensing the *idea* of a ratio. They shift decimal points in specific cases without knowing why. They multiply particular figures without thinking about multiplication as a process. They have been too much concerned with small skills to see relationships.

And they are what we make them. As adults, many of them shrink from number and avoid as far as possible all situations involving number. Their drill in school produced no favorable attitude toward it.

The meaning of number is a mathematical question. Our drill theory should certainly embrace this field. It begins when the child enters school and it ought to continue as long as he attends. First we have the concepts of small numbers and at the other end of a long course we have the Theory of Number.

In the primary grades it is all but fatal not to spend time in drilling upon the two aspects of number concepts—their reproduction and identification. Enough has been written elsewhere on this aspect of the question. Then, too, the different ways of knowing a number must be cared for—the series idea, the component idea, and the ratio idea.

As a rule the children of the primary grades will be working on the meaning of numbers at three levels simultaneously. The first level will be the one-place numbers, and these will be treated according to the suggestions contained in the last paragraph. The second level will be the numbers from 10 to 19, or the teens numbers. These will receive (in theory) all the treatment that was used on the first level plus the new ideas which have to do with these two-place numbers—the one 10 with or without accompanying units. (The zero will require some consideration.) The third level runs from 20 to 100 and utilizes (again in theory) all the earlier procedures. It permits some generalization as to the number system.

Throughout the middle grades the number system will be carried far enough—probably to billions to permit a general conception of the decimal system of whole numbers to be formed. Meanwhile decimals, beginning as money numbers perhaps as low as grade three, will be introducing a still wider generalization as the system is extended to the right of the one's place.

All this time a system of numbers not very closely related to the decimal system has been under development, namely, our system of common fractions. At this point most teachers and most courses give up the notion of imparting meaning—and this in spite of the fact that common fractions are older in the development of the race than decimals and are better known to children from their life experience. Many a child on entering school knows something about $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, and $\frac{2}{3}$; but no such child knows anything about 0.5, 0.25, 0.75, $0.33\frac{1}{3}$ or $0.66\frac{2}{3}$. While confining computational practice for the most part to fractions of small denominators, the meaning of fractions—which also requires practice—should suffer no such limitations. Within the scope of the system there should be the fullest opportunity to reach generalizations.

THE SOCIAL SIGNIFICANCE OF ARITHMETIC

There are some who seem to believe that if we instruct and drill on number in its mathematical meaning we shall have little else to do. For example, it is held that we shall thus fully reveal the meaning of the four fundamental processes. This theory is attractive but it is only partly true. Its truth lies in the fact that our present algorisms are all of a piece with our Hindu-Arabic system and are therefore explained by it. If the system is well learned, the algorism is half learned before it is begun.

This, however, is not the whole story. The algorism which we now use in multiplying is not multiplication; for other and quite different algorisms not now in vogue are also multiplication. The rest of the truth is that the essence of multiplication is outside our number system and that it existed before our number system was known.

What then will contribute to additional understanding of arithmetic? Undoubtedly its social significance. As for multiplication, it is all about us. It confronts us in tile and brick and glass. It is wherever there are rows and columns. It is found in every multiple purchase and wherever numbers are repeated. It is implicit in every rectangle and in every three-dimensional container. We live in a world of multiplication.

To bring this and similar significancies in arithmetic to the consciousness of the child and to do so frequently (the drill idea) is to contribute to his sense of the value of the subject, to the uses that he will make of it, to his favorable attitude toward it, and to his understanding even of its mathematical meaning.

There are so many social uses of arithmetic that one cannot name them. They are often institutional; for society, having uses for number in certain special ways, has slowly created the necessary organizations. The insurance company is a good example. In that case number is used to create and maintain a collective effort where small sacrifices are pooled to prevent large losses or to afford one large benefit at a time of crisis. The social significance of insurance is the important matter, not the computation involved in it.

Other aspects of the social significance of number are not so sharply institutionalized. For example, we have an arithmetic of production, of distribution, and (quite consciously) of consumption, yet we should hardly say that production, distribution, or consumption are organized to the point where we can call them institutions quite as we can our system of banks or our system of insurance companies. Among these quasi-institutionalized areas, that of consumption is especially attractive, partly because we are all consumers and partly because the field of consumer arithmetic has already been worked to some extent.

Finally, the social significance of arithmetic concerns the individual—as a member of society. Daily, almost hourly, we need arithmetic in the reading of newspapers, magazines, and books. We hear arithmetic and we talk arithmetic. Arithmetic is our daily need—and not ours alone. It is also the need of young children. How is it that they have learned so much of this highly artificial system in the short six years before they enter school? They have had no human instruction, except perhaps in rote counting, because the home, if it teaches the child any school subject, teaches him reading. Yet careful investigators such as Nila B. Smith and Marion J. Wesley report an amazing use of number on the part of six- and seven-year-old children. They count; they add and subtract; they multiply and divide; they use fractions; they measure and compare. They have actually begun nearly all the things they will ever do in arithmetic.

If the lives of first grade children are so full of number, then the lives of older children and of adults are, of course, still more so. In this world of science, of the machine, and of instruments of precision those who are quantitatively intelligent find large and varied uses for arithmetic in achieving their many purposes. All of this points to the *social significance* of number and to the need for its recognition in our educational program.

It is worth while, as we bring to a close this treatment of the field of drill, to say that not meagerly, nor grudgingly, nor merely on the occasion of an "activity," but fully, freely, and systematically, the social significance of arithmetic should find a place in our program. The aspects of it will vary more than those of any other topic we have discussed. We are now, therefore, upon the outer edges of the field of drill. Our repetition has more variety and less identity than ever before. If it were worth while to arrange the types of material to which drill may be applied from those in which repetition was most obvious to those in which it was least obvious, the first types of material would be computational and the last would be social.

Yet, though near the boundary, we are still within our proper area. Repetitive elements may be subtle, but they are nevertheless real. It is still true that recurrence of experience fosters learning. There is still a place for drill.

The title of this chapter poses the question "What Becomes of Drill?" To which one may reply, "Why should anything become of it? Why is the question raised?" It is raised because of the rapid change in the curriculum and the objectives of arithmetic during recent years. In fact, as the reader of this Yearbook will readily infer, arithmetic is on the march. It is plainly evolving from a narrow computational subject to a subject of broad social and mathematical import. The great incentive that used to be offered to boys to do their "sums" was that business needed fast and accurate computers, and that jobs would await them if they could qualify. Much of this has passed. Fast and accurate computers are no longer men but machines. Meanwhile—partly because of the machines—the use of numbers in social ways has been enormously increased. The youngster of today must know what these numbers mean—their mathematics. He must also know what they signify as indicators of social conditions. Arithmetic—that is, arithmetic as used in the modern world—has spread from the counting house into the general life of the people. What then becomes of drill? If arithmetic is indeed "on the march," then drill should follow the flag. If in the long fight against ignorance it has served well, it must not remain in the old camp with the rear guard.

Chapter X

THE EVALUATION OF LEARNING IN ARITHMETIC

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For the best interests of both, we have sometimes been told, instruction and measurement must be kept apart. Considerations involving measurement are held to impede the solution of problems of measurement; and, conversely, the task of measurement is thought to require time and energy which teachers might more properly give to instruction. Some test technicians have not felt particularly handicapped by their ignorance of the purposes of arithmetic instruction. On the other hand, teachers, who supposedly know the nature and purposes of their subject matter, are regarded as unable to evaluate the learning they have directed.

This separation between measurement and instruction presents a curious anomaly. It exists only in theory; it cannot actually be maintained in practice; hence it is wholly artificial. Nevertheless, the attempt to establish the separation has been made, and the effects upon classroom evaluation and teaching alike have been most unfortunate. Let us start with these unfortunate effects, reserving for a moment consideration of the artificiality of the separation.

Effects of attempted separation of measurement and teaching. One effect of the attempted separation has been to remove measurement further and further from the immediate learning situation. Tests to be used for diagnosis and the evaluation of achievement have been standardized, and in the process of standardization have lost touch with the features peculiar to the local classroom. Indeed, it is the very essence of standardization as a process to disregard local variations and to strive for a "national average" of subject content and teaching practice. For certain purposes, such as are comprehended in the survey function of measurement, for example, this averaging and this disregard for local conditions are precisely what is needed. For other purposes, however, especially for those purposes which relate most closely to the organization and direction of learning, it is the local conditions, lost in standardization, which are most crucial.

A second effect of trying to keep measurement and teaching apart has been to limit measurement to outcomes that can be most readily assessed. In arithmetic this has meant concern almost exclusively with "facts," with computational skills, and with "problem-solving" of the traditional sort. The necessity for developing proficiency in these areas is obvious. Furthermore, measurement of these outcomes (or at least partial measurement) can be managed by objective techniques. As a result, many experts in test construction have confined themselves for the most part to these outcomes, as have also teachers who have followed their lead. But, as will be made clear below, there are other arithmetic outcomes, fully as important if not so obvious as those commonly attended to, and these outcomes under present practice are neglected.

A third ill effect of the effort to separate measurement from teaching has been to limit unduly the techniques which are serviceable for evaluation. It is not far from the truth to say that evaluation in the broad sense has been narrowed to measurement in the narrow sense, that such measurement has been made virtually equivalent to testing, and that testing has become the administration of paper-and-pencil tests, usually of an objective character. Valuable as are such objective tests, whether commercial or local products, they cannot readily carry the full burden of evaluation. There are other procedures which are now ignored. These other procedures, to be described below, are easily managed by teachers and, what is more important, they uncover kinds of learning processes and products which at present elude paper-and-pencil tests.

The fourth harmful effect of the desire to separate measurement and teaching, and the last one here to be considered, has been to create confusion with respect to the purposes of evaluation. Learning may be evaluated for a number of reasons. One may be the diagnosis of failure; another, the measurement of progress over short units or sections of content; yet another, the pre-testing of abilities before starting a new topic, as a means of "establishing a base line" or determining a common ground for instruction in a given grade. Tests, commercial or otherwise, which are suited to one purpose are not thereby suited equally well to other purposes. Indeed, it might even be argued that the better a test serves one function, the less well it can serve others. For illustration, the better a test "surveys" (e.g., permits comparisons of a school system with other school systems or with national norms) the less effectively it "diagnoses." Yet, several survey tests in arithmetic incorrectly bear the label "achievement test" and are improperly used for the latter purpose, as well as for other purposes even more remote from the function for which they were devised.

The four consequences of the attempt to divorce measurement from instruction which have been mentioned reveal the essential artificiality and impracticability of the attempted separation. The fact of the matter is that, despite the effort to do so, measurement and teaching have not been kept apart. Classroom instruction has emphasized the outcomes and only the outcomes with which measurement technicians and some measurement theorists have worked, and it has used the evaluation procedures and only the evaluation procedures favored by them.[1] The poverty of the results of arithmetic teaching under this scheme of things has been all too well exposed. It has been exposed in the fact that children are not able to use the arithmetic they are taught in dealing with the simple personal quantitative problems of the adult world into which they are presently to enter; indeed they are not even sensitive to the quantitative aspects of their own daily lives. The poverty of our teaching has been exposed, too, in the arithmetical deficiencies of adults who have been subjected to eight or more years of school arithmetic. Witness the typical adult's efforts to evade the quantitative demands of his life and his embarrassment and uncertainty when there is no escape from them.

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[1] There are of course occasions when measurement and instruction must be separated. This is true when tests are correctly used in some kinds of research and for survey purposes. It is true at times when pupils are to be classified for certain reasons, as, for example, in making up sections in a cosmopolitan junior high school which draws from many lower schools. But this chapter deals with the evaluation *of learning;* in other words, with evaluation as it relates to the improvement of teaching. On this account the consequences of separating measurement from instruction, helpful in non-instructional situations, are here viewed as harmful.

To some readers the foregoing discussion may appear to be somewhat academic and unreal. Among these readers will be the many excellent teachers who never have thought of separating measurement from teaching but have steadfastly evaluated learning in order to improve the effectiveness of their teaching. Admittedly, the past few pages have not been written for such persons. Instead, they have been intended for persons who have not yet accepted the full import of modern conceptions of education according to which children and the problems relating to their learning are the central consideration in the classroom.

The purposes of this chapter. The purposes of this chapter are two in number, the first of which has been already foreshadowed in the criticisms offered in the paragraphs above. The purposes are: (1) to outline a point of view with respect to evaluation (*a*) which relates evaluation to teaching, and (*b*) which makes evaluation comprehensive enough to include all objectives or aims of arithmetic instruction; and (2) to illustrate as far as possible concrete procedures for evaluating outcomes now too often overlooked. In order to save space for the second and more obviously practical purpose, the statements to be made with regard to the first purpose must be held to what is essentially an outline.

A POINT OF VIEW WITH RESPECT TO EVALUATION

As has already been suggested, the thesis of this chapter is that instruction and the evaluation of learning cannot be kept apart in theory and should not be kept apart in practice. Instruction and evaluation go hand in hand. As teachers develop new insights into learning—its difficulties, its stages or phases of development, the basic understandings required for each advance step in learning—as teachers acquire these insights, they will employ them in improved evaluation. And as they correct or modify their evaluations and devise procedures which are more comprehensive and more penetrating, they should come upon new data of great significance for the better guidance of learning. Viewed thus, instruction and evaluation are inseparable and mutually interdependent.

If the recommendations to be made in this chapter were accepted and practiced, certain values would accrue. In the first place, evaluation (the broader term will henceforth be used in place of measurement) and teaching would both start with arithmetic outcomes—with *all* the arithmetic outcomes which are deemed worthy of attainment.[2] Evidences would then be obtained, so far as can reasonably be done, on all types of growth and at all stages in this growth. In the second place, any procedure whatsoever which might shed light on learning would be accepted and utilized. Wider recognition would be given to observation, to the interview and conference, and to other techniques which can yield information on the progress of learning. In the third place, the various purposes for which evaluation is undertaken would be recognized in the kinds of procedures which would be used. In other words, evaluation procedures would be adapted to the ends for which they are best fitted. In the fourth place, evaluation would be continuous with teaching and learning. Evidence on learning would be collected daily (occasionally by tests, more often by informal procedures), instead of irregularly and spasmodically. In the fifth place, evaluations would be immediate and intimate; they would reflect the unique conditions, emphases, and factors affecting learning in particular classrooms. And for this reason they would provide teachers with the kinds of information which they most need in order to direct learning.

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[2] It is no longer possible to assume that a test of computational or problem-solving ability measures all arithmetical outcomes—if indeed this assumption ever was tenable on logical grounds. Brueckner, using Vocabulary and Quantitative Relationships sections of the Unit Scales of Attainment with 453 children in grades four A to five B, obtained coefficients of correlation of .361 between "vocabulary" and "computation," and of .522 between the former and "problem solving." Scores on the Quantitative Relationships test correlated .576 and .661 with "computation" and "problem solving," respectively. (See Leo J. Brueckner, "Intercorrelations of Arithmetical Abilities," *Journal of Experimental Education*, 3 : 42-44, September, 1934.) Since that date, Spainhour's investigation has confirmed these results. Spainhour devised special tests of "mathematical understanding" (reliability coefficients of .901 for grade four and of .933 for grade six), and administered them along with the New Stanford Reasoning Arithmetic Test and the New Stanford Computation Arithmetic Test to 143 children in grade four and 136 in grade six. In grade four the "understanding" test scores correlated .665 and .751 with "problem solving" and "computation," respectively. In grade six the corresponding *r*'s were .751 and .756. (Richard E. Spainhour, "The Relationship Between Arithmetical Understanding and Ability in Problem Solving and Computation." Unpublished Master's thesis in education, Duke University, 1936.)

These gains are not to be won by wishful thinking. Admittedly, they represent an ideal state of affairs which now can be no more than approximated. But even so, the approximation in itself will mean important progress away from present practice in evaluating outcomes.

Comprehensive and functional evaluation in arithmetic is dependent upon the effective relating of a number of factors. We need, first of all, an adequate statement of arithmetic outcomes. Second, we need to recognize the peculiar demands of different purposes in evaluating learning as these demands affect evaluation procedures and instruments. And, third, we need to have a practicable program for evaluation. This third requirement involves *(a)* knowledge of effective procedures and their limitations, *(b)* a realistic understanding of what can and cannot be done by the classroom teacher, and (*c*) the actual planning and designing of evaluation procedures. This last named item (*c*), is reserved to the last part of this chapter. The other items listed above will now be considered in order.

Acceptable outcomes. There are dangers in setting arithmetic outcomes. In the first place, there is the danger that the mere listing of outcomes separately may for some carry the implication that these outcomes are isolated from each other and are to be achieved independently of each other, one at a time. But outcomes are not thus distinct; rather they overlap in meaning, and they are probably best achieved when they are achieved more or less together in their functional relationships. In the second place, there is the danger that outcomes may be regarded as ends to be attained once and for all, the quicker the better. But outcomes are not so to be conceived; they really stand for directions of growth. In the third place, there is the danger that from a list of outcomes questionable teaching practices and forms of curriculum organization may be inferred. Not always should particular outcomes be at the immediate forefront of thinking when learning activities are being considered. After all, as we have been told almost too often, we teach "the whole child." Outcomes function best in teaching when they constitute the *background* of thinking, against which may be projected possible teaching plans. When, however, evaluation rather than teaching is the major concern, outcomes as such become more immediately important. In the fourth place, there is the danger of a suggestion of finality in any formal statement of outcomes. These outcomes work their way into educational thought; they come to be accepted as definitive, and they are modified only with great effort.

In full awareness of these dangers a list of arithmetic outcomes is given below. This list is not the result of Committee action; it does not bear the label of the Committee's authority. Instead, it is offered purely as the writer's own formulation. After all, if evaluation must start with outcomes, we must know what the desired outcomes are, and the list below seems to the writer to be adequate at present as a working basis.

(1) Computational skill:

Facility and accuracy in operations with whole numbers, common fractions, decimals, and per cents. (This group of outcomes is here separated from the second and third groups which follow because it *can* be isolated for measurement. In this separation much is lost, for computation without understanding *when* as well as *how* to compute is a rather empty skill. Actually, computation is important only as it contributes to social ends.)

(2) Mathematical understandings:

a. Meaningful conceptions of quantity, of the number system, of whole numbers, of common fractions, of decimals, of per cents, of measures, etc.

b. A meaningful vocabulary of the useful technical terms of arithmetic which designate quantitative ideas and the relationships between them.

c. Grasp of important arithmetical generalizations.

d. Understanding of the meanings and mathematical functions of the fundamental operations.

e. Understanding of the meanings of measures and of measurement as a process.

f. Understanding of important arithmetical relationships, such as those which function in reasonably sound estimations and approximations, in accurate checking, and in ingenious and resourceful solutions.

g. Some understanding of the rational principles which govern number relations and computational procedures.

(3) Sensitiveness to number in social situations and the habit of using number effectively in such situations:

- *a.* Vocabulary of selected quantitative terms of common usage (such as kilowatt hour, miles per hour, decrease and increase, and terms important in insurance, investments, business practices, etc.).
- *b.* Knowledge of selected business practices and other economic applications of number.
- *c.* Ability to use and interpret graphs, simple statistics, and tabular presentations of quantitative data (as in study in school and in practical activities outside of school).
- *d.* Awareness of the usefulness of quantity and number in dealing with many aspects of life. Here belongs some understanding of social institutions in which the quantitative aspect is prominent, as well as some understanding of the important contribution of number in their evolution.
- *e.* Tendency to sense the quantitative as part of normal experience, including vicarious experience, as in reading, in observation, and in projected activity and imaginative thinking.
- *f.* Ability to make (and the habit of making) sound judgments with respect to practical quantitative problems.
- *g.* Disposition to extend one's sensitiveness to the quantitative as this occurs socially and to improve and extend one's ability to deal effectively with the quantitative when so encountered or discovered.

Purposes of evaluation. Evaluations of learning are undertaken, or should be undertaken, for several different reasons. These differing purposes introduce variations into the procedures and instruments to be used. For example, as the learning area which is sampled is extended, the thoroughness of the sampling diminishes. What seem to the writer to be the five chief purposes of evaluation are listed below.[3] In each instance both the meaning of the purpose and the significance of that purpose for the kind of evaluation procedure to be used are illustrated in terms of paper-and-pencil tests. Tests are employed in this connection because teachers are more familiar with them than with [illegible] of the other procedures. Illustrations of other evaluation procedures will be given later with greater emphasis.

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[3] Evaluation is also used to motivate learning, and this purpose of evaluation might have been included as the sixth discussed. This purpose was, however, omitted, first, because motivation is a general effect of evaluation regardless of the particular purpose for which it is used, and second, because abuses easily arise from this use of evaluation. For instance, when tests are used for this purpose, the motivation is likely to be extrinsic and harmful to sound learning, rather than intrinsic and beneficial.

The chief purposes of evaluation are:

1. *To Diagnose Class and Individual Difficulty.* Effectiveness of diagnosis is dependent upon the intensity or depth of evaluation. That is to say, diagnostic data increase in value as they pass from a mere locating of the places of difficulty to an analysis of the causes of difficulty. On this account, special care must be exercised not only to include (*a*) all critical steps, processes, uses, opportunities, and so on, and (*b*) enough samples of each to yield reliable measures, but also (*c*) to discover insofar as is possible the thought processes which children employ in dealing therewith. Diagnostic tests must usually be restricted in their range of coverage in order to insure such depth and intensity.

2. *To Inventory Knowledge and Abilities.* Inventories are made, for example, to determine "readiness" for a new topic or for the start of instruction in a grade. The usefulness of this kind of evaluation has been made steadily more apparent as we have sought better to adapt the pace of instruction to level of pupil ability. The inventory test resembles the diagnostic test in that it includes samples of all levels of thinking which are achieved in the critical aspects, or skills, etc. Its range, obviously, varies with the area under examination from a relatively few constituent abilities, thought processes, and evidences of social awareness with respect to basic skills and concepts (when "readiness" for a new topic is under consideration) to the many such aspects of skills and concepts of a whole year's work (when, for example, the "base line" for class instruction in grade four is being established). Usually the depth of evaluation is much less in an inventory test than in a diagnostic test, or it may justifiably be made less.

3. *To Determine the Extent of Learning over a Limited Period.* An example is the measurement of learning over a month's unit of work. When tests are used for this purpose, they may be called "instructional" or "progress" tests. They are commonly short enough to be given in one class period or less, and their content is restricted to what has been but recently taught. In this respect they tend to differ from inventory tests. They differ from diagnostic tests in that usually but comparatively little effort is expended

to get at the source of difficulty. But, as is also true in the case of the first two purposes, this third purpose of evaluation does not necessarily require a test at all. Indeed, in the case of some kinds of learning other procedures are much more suitable.

4. *To Measure Learning over a Relatively Long Period.* One use of evaluation for this purpose is to secure a basis for pupil classification; another, to get a general view of achievement for a semester or year to be used in making up the term "mark." Usually paper-and-pencil tests are employed exclusively for this purpose. Such tests are certainly valuable, but evaluation should not be limited to them since probably not all outcomes will be represented in them. Here, that is for evaluating learning over fairly long periods of time, the extent of the area is relatively much larger than in "progress" tests, and thoroughness of sampling must be sacrificed to some degree. As in the case of progress tests (Purpose 3) test content should be restricted to what has been taught in the period covered, except of course as skills, concepts, etc., taught earlier are indirectly involved. The purpose of "achievement" testing almost necessarily precludes diagnosis, save in the shallowest sense of finding places of difficulty, and by the same token it cannot, unless the test is unusually comprehensive, well serve the purpose of inventorying.

5. *To Obtain Rough Measures for Comparative Purposes.* Reference here is to the survey function of evaluation, as this is illustrated in comparisons between schools within a system or between school systems. Because of the purpose for which the measures are to be used, these measures must almost necessarily be obtained from paper-and-pencil tests. Objectivity in scoring and the reliability of measures are especially important. Unless comparisons are to be made grade for grade and on the subject matter of those grades, a single test for all grades is commonly used. This means that the learning area is very extensive, and that the sampling is correspondingly very crude. To use a test designed for the survey function for the purpose of achievement testing (Purpose 4), or for progress testing (Purpose 3), or for inventorying (Purpose 2), or for diagnosing (Purpose 1), is almost certain to introduce errors and to lead to misinterpretation of test results. Yet, precisely this practice frequently obtains in testing programs.

These five purposes of evaluation have been discussed in terms of tests and apparently in terms only of computational skills. But actually the points made apply equally well to other evaluation procedures and to all outcomes. The nature of other evaluation procedures is treated in the next section. Emphasis needs now to be given to the fact that in evaluating growth or learning toward all objectives, and not merely toward computational proficiency, one must keep in mind the purposes for which evaluation is undertaken. Thus, one needs to diagnose, to inventory, to note progress, etc., in the case of mathematical understandings and of sensitiveness to the quantitative in life, just as fully as in the case of computational skills. Growth takes place in understandings and in quantitative sensitiveness as truly as in computation, and just as truly learning in these areas needs to be assessed for difficulties, for status, for progress, etc.

Evaluation procedures. Four general classes of evaluation techniques will be briefly considered, namely, (1) paper-and-pencil tests (or simply tests), (2) teacher observation, (3) individual interviews and conferences with pupils, and (4) pupil reports, projects, and the like.

1. *Tests.* Though much of the preceding section has been devoted to the use of tests in evaluation, actually far more space than this whole chapter affords could be taken for this purpose. Of all the possible evaluation procedures no other has received the research and theoretical attention given to tests. With the limited space here available, only a few suggestions can be offered.

Valuable as tests are, they are subject to certain limitations: (*a*) Tests are essentially artificial: they are convenient substitutes for the ultimately valid trial by functional use. This is but another way of saying that the true measure of learning is the ability to use what has been learned in practical life situations. (*b*) Tests yield scores which are susceptible to errors of interpretation. We may infer, for example, that the score is a measure of the total ability tested, whereas it is but a description of a particular performance in a particular situation. Obviously, the more comprehensive our testing and the nearer our testing situation approximates functional use, the fewer will be the errors of interpretation. (*c*) Tests may be used too exclusively. We may rely on tests to furnish indirectly evidence of growth in areas to which tests are not very sensitive. It is exceedingly difficult to prepare tests (other than essay tests) to measure certain arithmetic outcomes. If tests alone are used, these other outcomes will almost certainly be excluded from evaluation. (*d*) In the attempt to secure reliable measures from tests, we may pay too high a price for objectivity in scoring. Consideration of such a factor as correct principle, or evident understanding, introduces variation in judgment and so makes for unreliable measures. If we hold to the superior value of reliable measures, we may lose much that we need in order to understand children's work habits and thought processes. If we hold to the superior value of these evidences of learning we tend to lose objectivity and reliability.

Another series of cautions relates to standard tests and the use of norms for such tests. It cannot be assumed that mere standardization of a test makes it a good test, even for the purpose for which it is designed: (*a*) The content may be quite different from what has been locally taught,[4] and the norms may therefore be useless in the local situation (except for the purposes of the survey). (*b*) It has been shown that the arithmetical content of such tests varies considerably.[5] (*c*) It is not unlikely that many, or most, standard tests measure effects of arithmetical learning different from the achievement and abilities with which teachers are primarily concerned. (*d*) Moreover, norms on supposedly comparable tests have been found not to be equivalent.[6] (*e*) Still again, nation-wide norms are apt to be misleading. Thus, nation-wide norms are worse than useless in diagnostic tests. They are of little or no service in inventory tests, except possibly to ascertain "readiness" for some topic. Certainly when we seek to determine where to start instruction in a grade, what we need to know is not how our pupils compare with other pupils but precisely what they can do about the items of skill, knowledge, and understanding which are regarded as prerequisites. And last of all, norms are of little value in progress or achievement tests unless we follow slavishly the instructional organization in terms of which the test has been standardized. Norms come into their largest and most legitimate field of usefulness in connection with the survey function of testing.

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[4] Guy M. Wilson, "Choosing and Using Standardized Tests in Arithmetic," *Education*, 40 : 177-179, November, 1939.

[5] Louise Beattie, "Standardized Tests in Arithmetic," *Educational Method*, 16 : 175-176, January, 1937.

[6] Stokes and Finch in 1932 reported a difference amounting to .77 grade between the medians of 65 pupils in grades seven to nine on the New Stanford Computation Arithmetic Test and the Van Wagenen Revision of the Woody Arithmetic Scales of Fundamental Operations in Arithmetic, and a difference of .74 grade in class medians on the New Stanford Reasoning Arithmetic Test and the Buckingham Scale for Problems in Arithmetic. (C. N. Stokes and F. H. Finch, "A Comparison of Norms on Certain Standardized Tests in Arithmetic," *Elementary School Journal*, 32 : 785-787, June, 1932.)

The results of this study have since been confirmed by Foran and Loyes (Thomas G. Foran and Sister Mary Edmund Loyes, "Relative Difficulty of Three Achievement Examinations," *Journal of Educational Psychology*, 26 : 218-222, March, 1935), and by Pullias (Earl V. Pullias, *Variability in Results from New-Type Achievement Tests*, Duke University Research Studies in Education, No. 2, Duke University Press, 1937. See especially Chapters VII and VIII.)

What has been said about tests is deliberately negative. These negative comments imply no doubt of the values of tests for evaluating learning. The shortcomings and limitations of tests have not been stressed in any attempt to get rid of them. We should be in a sad state indeed if tests, local, commercial, or both, were abolished. To say that tests will not do the whole job of evaluation, and to say that tests need to be constructed and test scores interpreted with reasonable caution is far from saying that tests should be abandoned. Rather, they should be retained and improved, and they should be supplemented by other evaluation procedures. The import of this whole section may be summarized in the statement that tests usually indicate the outward, objective, quantitative results of learning, not the inward, subjective, qualitative results; yet these latter both transcend and include the former.

2. *Teacher Observation.* The type of evaluation meant here is the type which the intelligent and alert teacher uses daily in analyzing and assessing the written and oral work of his arithmetic pupils. The observation referred to may be informal, as just described, or it may be more closely controlled, as when special settings are arranged in order to note children's behavior with respect to the quantitative aspects of their experience. An excellent example of informal observation will be found in Professor Wheat's chapter (pages 80-118). Other examples will be found on pages 259 and 260 which follow.

The possibilities of evaluating through observation are wellnigh limitless. That these possibilities have not been realized is explicable on two grounds. In the first place, teachers have had their confidence in their own judgment undermined by certain research which purportedly has demonstrated its unreliability. The implications of this research are that teachers' estimates and opinions are of doubtful worth and that they must give way to more trustworthy (that is, more objective and reliable) evaluation techniques. Some of the dangers inherent in this view have been pointed out,[7] and eventually the artificiality of the techniques in this research will be disclosed. In the meanwhile evidence is accumulating that teachers' judgments (in the form of scores on essay tests, of marks, and the like) need not be as unreliable as had been supposed.[8] But the reaction against the extreme position of the objectivists has not yet affected teachers to any large extent. Under the influence of the objectivists teachers tend to doubt the validity and reliability of their own observations and to minimize the usefulness of observational data for the purposes of evaluation. Nevertheless, it is probable that 90 per cent of teachers' (*good* teachers) activities in teaching and evaluating are still subjective and must remain so. If this be true, the wise course would seem to be to help teachers to get the most from their observations, rather than to continue to discourage their use.

A second obstacle to confident use of observation is of a different character. To observe accurately one must know what to look for. Teachers who regard arithmetic purely as a skill subject are hardly equipped to make useful observations of growth in arithmetical insight and quantitative sensitiveness. On the other hand, teachers who are committed to a more modern conception of arithmetic are handicapped by our general ignorance concerning learning. We do not yet know with certainty all that we need to know about growth toward the non-computational aims of arithmetic. As new information of this kind is acquired, and as teachers regain confidence in the value of observational procedures and in their ability to use this method, evaluation by observation will become more precise and more comprehensive.

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[7] W. A. Brownell, "The Use of Objective Measures in Evaluating Instruction," *Educational Method*, 13 : 401-408, May, 1934.

[8] The following studies are illustrative:

Howard Easley, "On the Limits of Predicting Scholastic Success," *Journal of Experimental Education*, 1 : 272-276, November, 1933.

James B. Shouse, "College Grades Mean Something," *Journal of Educational Psychology*, 30 : 510-518, October, 1939.

Ralph R. Wolf, Jr., *Differential Forecasts of Achievement and Their Use in Counseling*. Psychological Monograph, No. 227. American Psychological Association, Inc., Columbus, Ohio, 1939.

The advantages of evaluation by observation (only skillful observation, of course) are several in number: (*a*) Observation has few of the limitations of testing as to time and place. Informal observation requires no special planning and no special arrangements: one can seize each instance of significant behavior as it occurs. (*b*) Observation "catches" behavior in all its functional relationships. One sees not only the error, for example, but also the prior and the accompanying behavior which may account for the error. (*c*) Evidence for evaluation is obtained when it can be used. This is particularly true in the case of diagnosis. (*d*) Observation imposes no unusual restrictions and exposes children to no unnatural tensions. (*e*) Evaluation by observation enables teachers to secure evidence with respect to many arithmetic outcomes to which testing is ill adapted. Reference here is to the outcomes listed especially under mathematical understandings and quantitative sensitiveness on pages 232-233.

3. *Individual Interviews and Conference.*[9] As its name implies, this evaluation procedure, unlike observation, is limited to contacts with individual children. The teacher not only observes what the child does, but has him "talk out loud" as he works and questions him whenever the oral report is interrupted, is incomplete, or is ambiguous. The purpose is primarily to get at the way in which the child thinks about the given quantitative situation, be it a computation, a verbal problem, the use of technical terms, the interpretation of the number relations, etc. The usefulness of the conference for probing undesirable attitudes (indifference or open dislike) toward arithmetic, for finding out how extensively a child really uses arithmetic, and the like, should be apparent. Examples of the interview will be found on pages 259 and 260.

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[9] Observation and the interview are not of course as unrelated as may be suggested from the separate treatment accorded them here. As a matter of fact, the interview multiplies opportunities for observation and makes these observations more intimate and penetrating.

Readers of research are already familiar with the use of the interview for discovering children's work habits in dealing with number combinations and computation.[10] Good teachers have always employed the interview and conference to some extent, chiefly for purposes of diagnosis. No other procedure equals the interview in disclosing the nature of disability and thus in providing data for remedial instruction. If this procedure were used more commonly in connection with initial instruction, later diagnosis and remediation would be greatly reduced in amount.

Perhaps the chief reason why the interview is not more generally used is the fact that it is time-consuming. To take a fourth grade child through the whole addition section of the Buswell-John Diagnostic Test requires between twenty-five and forty-five minutes. It is only rarely, however, that such extensive interviewing is necessary. Five or ten minutes can hardly be spent more profitably than in interviewing a child who is having trouble at some point.

The interview and conference should be kept flexible. Questions cannot be standardized, but must be worded and reworded until the child knows precisely what is required of him. At the same time the interviewer must avoid giving cues and must not ask leading questions. The child who is in difficulty gropes for reasons and explanations and is especially likely to seize upon any suggestion from the interviewer which may even temporarily rescue him from his predicament.

4. *Pupil Reports, Projects, and the Like.* Under this heading belong a large variety of pupil activities which are useful for teaching, as Miss Sauble points out (pages 181-195), but equally useful for evaluation. Individual children or groups of children can prepare reports on such topics as: How Number Is Used in the . . . . Bank, How We Got Our Figure 7, Lucky Numbers, The Arithmetic I Use on My Paper Route, Where Our Measures Came From, How Much Our Automobile Costs Us a Month,

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[10] For example: Guy T. Buswell and Lenore John, ***Diagnostic Studies in Arithmetic***, Supplementary Educational Monograph, No. 30, Department of Education, University of Chicago, 1926; Lofton V. Burge, "Types of Errors and Questionable Habits of Work in Multiplication," *Elementary School Journal*, 33 : 185-194, November, 1932; Lenore John, "Difficulties in Solving Problems in Arithmetic," *Elementary School Journal*, 30 : 675-692, May, 1930.

Shortcuts in Addition, The Budget for Our Camp Last Summer, and Number Tricks. Such reports are teaching devices, to be sure, but they also reveal unmistakably how sensitive children have become to the mathematics of number and to the quantitative aspects of life about them.

Trips and excursions to various points of interest offer other occasions for detecting awareness of the quantitative. Children who during the trip note and afterward can describe many uses of arithmetic are clearly more advanced in their appreciation of the social significance of number than are others who are less successful. Events in the history of number (e.g., the derivation of a standard unit for "foot") can be dramatized in such a way as to permit evaluation of the level of quantitative thinking which has been attained. In the primary grades the ability to act out or to picture the events in verbal problems reveals understanding of process meanings. The preparation of scrapbooks, models, posters, and special exhibits, all of them involving number and quantity, may be used as much to reveal the level which children have achieved in knowledge, understanding, and quantitative sensitiveness as to teach them new concepts, new skills, etc.

It is precisely the meanings, understandings, and appreciations of arithmetic that are the most difficult to evaluate objectively, reliably, and validly by means of tests. However, these outcomes become more susceptible to evaluation, evaluation that is none too objective and none too reliable, it is true, but nevertheless valid—when evaluation escapes the limits of testing and takes the form of one or another of the procedures described above. The peculiar advantage of special reports, dramatization, picturization, and so on is that they instigate behavior in which these outcomes appear in natural relations and functional reality. The loss in objectivity need not be serious. In the first place, it is better to have some evaluation, even if somewhat unreliable, than to have none at all. In the second place, the reliability of the final evaluation by these subjective procedures increases materially and attains a respectable figure if enough observations are made.[11]

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[11] This last fact is not sufficiently recognized. Assume that the reliability coefficient of a single observation is only .40. If the same kind of observation of the same performance and yielding the same result is made ten times, the reliability coefficient (Spearman-Brown formula) becomes .83. Fourteen observations would

One last word should be said about the opportunities for evaluation afforded by activity units and pupil projects in which number and quantity are involved. Too often the arithmetical aspects of these units are overlooked both by teachers and by pupils. The alert teacher, however, can detect these arithmetical aspects of units and projects, can direct the pupils' attention to them, and eventually can lead pupils to discover them for themselves. When this has been done, many occasions will arise when children will be able to use number in units and projects. Whether they do so or not, the *chance* for evaluation in either case is present and should be fully utilized. That this chance is not generally so utilized is regrettable, for units and projects afford splendid opportunities to observe children thinking mathematically when the stage has not been deliberately set, as it is in the arithmetic period.

Evaluation and the classroom teacher. The program of evaluation outlined in this chapter is obviously impracticable if it is inferred that every classroom teacher must find or devise and use procedures for all the purposes of evaluation and for evaluating growth toward all outcomes. Under this assumption the teacher would be so busy evaluating that he could never get around to teaching. It follows that judgment must be exercised in deciding the extent to which evaluation is to be undertaken and the manner in which it is to be carried out. In this connection the following steps may be suggestive.

1. The first step toward effective evaluation is to know and understand the outcomes set for instruction. In a given situation these outcomes may or may not be those listed previously on pages 231-232, but whatever they are, teachers, supervisors, and principals should know what the outcomes mean.

2. The second step for the classroom teacher is to know the various kinds of behavior which evidence growth toward these objectives and to train himself to detect this evidence. Reference here is primarily to kinds of growth which cannot be evaluated by means of tests (chiefly the outcomes listed under mathematical

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yield a reliability coefficient of .90. The writer recognizes at least some of the dangers in this statistical approach, but believes that the argument is nevertheless essentially sound.

understandings and sensitiveness to the quantitative). Admittedly, research has not yet identified all the significant evidences of growth which are needed for ideal evaluation, but this lack of research data should not greatly impede the teacher who understands what he is to teach in arithmetic and how his pupils learn.

3. Perhaps the third step is for the teacher to re-establish confidence in his ability to assess growth toward the more "intangible" outcomes. The values of observational procedures in the hands of the intelligent teacher have not been fully realized.

4. Closely related to the third step is the fourth: to take advantage of the close relation between teaching and evaluation and to seize every opportunity offered by everyday instruction to secure evidence of growth.

5. The fifth step, or another step, is to realize that evaluation for certain purposes (especially for purposes of measuring long-time achievement and of survey comparisons) is required rather seldom, and then may be managed by others than the individual teacher. This means that the teacher can concentrate on diagnosis, inventorying, and measures of short-time learning—on evaluation for those purposes, in other words, which bear most directly upon the major concerns of his pupils.

The values of observational procedures are emphasized in this chapter. To some, large use of these procedures may seem to put excessive demands on the limited time teachers have. Such is not the case; observational procedures are actually time-savers. A few moments devoted to careful observation at the critical time are worth more than an hour of less intimate diagnosis at a later date. The more observational procedures are used, the less is the need for more time-consuming procedures (interviews, tests, and the like). As has been stated before, observation "catches" the behavior which is significant for directing learning at the crucial instant, and so can obviate or at least greatly lessen the necessity for more elaborate evaluation procedures later on.

PRACTICAL SUGGESTIONS FOR EVALUATION

The plan of treatment in this section is to consider the possible means of evaluation for each group of outcomes in the order in which they are listed earlier in this chapter, and then to supply samples of procedures in the case of less easily measured outcomes.

Computational proficiency. Insofar as computation is interpreted to mean the mechanical manipulation of numbers in solving abstract examples and verbal problems of the traditional kind, the outcomes in this area are probably the most amenable to evaluation. For this reason, and also because both testing instruments and critical discussions[12] are generally available, the problems of evaluation in this area are passed over briefly, and no sample procedures are given.

The fact that most computational outcomes seemingly can be evaluated so readily by means of tests has given rise to certain unwise practices. Perhaps chief among these is that of using local uniform supervisory tests and commercial standard tests to coach children. It is not unusual for teachers and administrative officers to purchase all forms of a given standard test (or for teachers to maintain a library of all supervisory examinations over a period of years) to train children systematically on the content and skills involved in taking these tests.[13] It should be obvious that under such conditions the test scores later obtained are practically useless for purposes of evaluation. Other evil consequences of this practice should be equally apparent.

1. *Diagnosis.* For the purposes of group diagnosis on particular arithmetic skills (*a*) some standard tests are useful. Here may be mentioned the Brueckner Diagnostic Tests (Whole Num-

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[12] In the writer's opinion the best reference relating directly to arithmetic is the chapter by Greene and Buswell in the *Twenty-Ninth Yearbook of the National Society for the Study of Education.* These authors employ a terminology somewhat different from that in this chapter, but the reader should encounter no difficulty in translating the one vocabulary into the other. See: Charles E. Greene and Guy T. Buswell, "Testing, Diagnosis, and Remedial Work in Arithmetic," *Report of the Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education, Part I,* Chapter V. Public School Publishing Co., Bloomington, Ill., 1930.

[13] A letter from the publisher of many educational tests, who must remain anonymous, states: "I could give you the names of several school systems in which cumulative files are kept of all forms of our tests. We have standing orders from these systems to supply them with each new form as it appears. Our agents tell us that in these systems the tests are available to all teachers who, if not encouraged to do so, are certainly not prevented from duplicating these tests and drilling their pupils in taking them. Then some form or other of these tests is used at the end of the year to measure achievement and to make comparisons between classes within the same system!"

bers, Fractions, Decimals) and the Compass Diagnostic Tests (twenty forms). These tests show the places of difficulty, but not the reason for difficulty or even at times the specific nature of difficulties. The same tests may be used with the same limitations for individual diagnosis. Similar statements may be made (*b*) about the group diagnostic tests sometimes provided in textbooks and manuals by textbook authors and (*c*) about group diagnostic tests constructed by the teacher or the supervisor. (In this connection attention is directed to the earlier discussion, page 233, of diagnostic tests.) Diagnosis is better, the closer it comes to the immediate learning situation. On this account more crucial data than those obtainable from group tests are to be secured (*d*) from observation and, in the case of individual children, (*e*) from personal interviews. The Brueckner Diagnostic Tests, mentioned above as useful for group diagnosis, may also be used for interviewing individual children, and the Buswell-John Diagnostic Tests for Fundamental Processes in Arithmetic has been especially devised for interviewing in connection with the addition, subtraction, multiplication, and division of whole numbers only. In the latter, the test for each operation requires an average of more than a half-hour per child. Since the content in each case includes the whole range of skills taught, the first and easiest items may safely be omitted in the higher grades, and the later and harder items, in the lower grades. Or, following the model set by the Brueckner and the Buswell-John tests, the teacher can prepare his own test for interviewing. But tests as such are not indispensable to diagnosis. On the contrary, from watching his pupils day-by-day as they are busy with their arithmetic work the teacher may gain his most valuable insights. The daily lesson provides the ideal time and place both for diagnosis and for remedial teaching. As has been repeatedly emphasized, the practice of careful and continuous observation makes not only for help at the critical time, but also for economy of effort.

2. *Inventorying.* To determine "readiness" for the work of a grade or for some particular arithmetic topic, class evaluation is most practicably undertaken (*a*) through the testing programs provided in certain textbooks and teachers' manuals, (*b*) through specially prepared tests constructed according to the local course

of study, and (*c*) through observation, particularly if cumulative records are kept. The last named, (*c*), does not, however, give an adequately general picture of the situation to aid in determining what should be done for the class as a whole. To the degree that instruction is individualized, however, evaluation should include observation (*c*) as well as (*d*) interviews whenever possible.*

3. *Short-Time Achievement.* For measuring attainment over a limited period of time or a single unit of work (*a*) textbook or manual tests, insofar as they fit the immediate situation, and (*b*) specially constructed local tests are to be preferred. The latter need to be carefully prepared; otherwise, important steps may be omitted. (*c*) Observation and (*d*) interviews serve supplementary purposes, which is to say that they are useful chiefly for evaluating outcomes which are important in the unit as a whole but are essentially non-computational. (*e*) Standard tests are of slight value, since rarely does their content agree with the local course of study.

4. *Long-Time Achievement.* For semester or yearly measures, the suggestions made with respect to short-time measures hold with equal force. The content of tests is chosen, of course, from a wider range than in the case of "progress" tests, since the learnings are wider in scope. Mistakes of interpretation are frequently made by obtaining measures of long-time achievement ("long-time" in the sense here used) from commercial standard tests which cover the arithmetic content of a series of grades.

5. *Surveys.* For comparisons between schools within the same system local instruments which are uniform in content may be used. These instruments, when intended for grade-by-grade comparisons, should probably contain only the content (skills, facts, etc.) taught in the given grade and in grades below that point. In this case as many separate tests may be needed as there are grades tested. Standard survey instruments are useful, particularly when the standings of school systems are to be compared with one another. In such cases the test contains a rough sampling of all skills, facts, etc., taught through grade three, grade six, grade eight, or some such point. Among the better known tests of this last

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* Since this chapter was written, Professor Brueckner has announced the publication of standard readiness tests for all processes with whole numbers, fractions, decimals and percentage. These tests are published by the John C. Winston Co.

kind are: Analytical Scales of Attainment in Arithmetic (grades three and four, five and six, seven and eight), Compass Survey Test (elementary and advanced), Metropolitan Achievement Test (primary, intermediate, and advanced forms), New Stanford Achievement Test (primary and advanced forms), Progressive Arithmetic Tests (primary, elementary, and intermediate forms), Public School Achievement Tests (Computation and Reasoning), and Unit Scales of Attainment in Arithmetic. For critical comments on these and other standard tests, the reader is advised to consult the various annual volumes under the general title *The Mental Measurements Yearbook,* edited by Professor Oscar K. Buros of Rutgers University. The non-equivalence of norms on survey tests, mentioned on page 236 above as invalidating comparison of scores for individual pupils, does not seriously affect the comparisons of large numbers of children or of schools as wholes. Nevertheless, the interpretation of test results should always include appropriate recognition of the implications of differences in aims and objectives among the schools concerned.

Mathematical understanding. This section presents illustrations of evaluation devices useful in connection with the less commonly assessed outcomes of arithmetic.

1. *Purposes of Evaluation.* So far as evaluation for differing purposes is concerned, outcomes which can be classified under the term "mathematical understanding" can be discussed together, since the problems relating to evaluation procedures are much alike for all of them. Exceedingly little has been done either informally or systematically to find practicable and valid procedures for evaluating the outcomes under the heading above. There are, for example, no standard tests available, except (*a*) two sections of the Analytical Scales of Attainment, one devoted to forty items testing "Quantitative Relationships," and the other to forty items testing "Arithmetic Vocabulary" (these sections being available in the tests for grades three and four, five and six, and seven and eight), and (*b*) a shorter section in the Iowa Every-Pupil Test. But these sections do not evaluate learning with respect to all the outcomes listed here under *Mathematical understanding;* they do not evaluate fully or for all different purposes with respect to any one outcome listed; nor do high scores on these tests guarantee

that the knowledge revealed will actually function to affect conduct. They are, however, to be recommended to anyone interested in objective means of evaluating in this area.

For diagnosis, the best procedures in the order of their present practicability are: (*a*) observation (both of oral and of written work), of which samples are given below, (*b*) the interview and conference, also illustrated below, and (*c*) specially constructed tests. The last-named are hardly practicable for the average teacher, for the preparation of such tests is as yet highly technical. Few ventures would, however, pay larger returns on effort than the cooperative work of a group of teachers in attempting to devise testing instruments. For their benefit what seem to be promising devices are illustrated below. Procedures (*a*) and (*b*) are entirely practicable for the teacher who knows what to look for. The present methods of preparing arithmetic teachers, however, hardly equip teachers with this kind of knowledge.

Inventorying to determine "readiness" is likewise handicapped at present because of inadequate knowledge of the stages of development which characterize growth toward the various objectives. All understandings are matters of experience, but the essential experiences cannot all be had at once. Rather, learning activities must be arranged so as to encourage growth from stage to stage, or from level of meaning to level of meaning. In time, tests should be available to identify these stages of development. In the meantime, the situation is not hopeless. Teachers (*a*) by observations and (*b*) by interviews can determine whether pupils have attained requisite degrees of understanding, and have or have not carried a given generalization far enough to warrant either its use in a new context or its extension through new activities. It is not impossible that (*c*) tests can be worked out experimentally and improved with trial. In this case, the sample items below can be adapted to the special purpose of inventorying.

Evaluation of learning over short periods of time is subject to the same limitations as have been mentioned for the other two purposes of evaluation which have been discussed—restricted knowledge and lack of procedures which are certainly reliable and valid. Reliance for the time must be placed upon subjective judgment, that is, on (*a*) observation and (*b*) interviews, chiefly the former. Eventually (*c*) tests may be available, and groups of teachers may want to try their hand in this direction.

What has just been said applies also to evaluation of learning over longer periods of time.

For survey purposes, survey tests are required. The only ones to be had have been mentioned in the first paragraph above under *Purposes of Evaluation*.

2. *Samples of Evaluation Procedures.*[14] Below are given samples of procedures (test items, observation, interview) which can be used for evaluating growth in mathematical understanding. Space limitations forbid more than a few samples in each case. The same limitations make it necessary to present each item as briefly as possible, without indication of its grade level and frequently without its being cast into proper form for testing.

TESTING DEVICES

The following testing devices are presented in the hope that teachers may be able to recognize the usefulness of the items and adapt them to their own needs and purposes.

(1) *The Meaning of the Number System:*

- *a.* Write the largest five-place number you can. . . . . . .
- *b.* Write the smallest five-place number you can without using 0. . . . . . .
- *c.* Suppose that the figures 3 and 8 are interchanged in the number 835. Would the new number be larger, or smaller, or the same size? . . . . . .
- *d.* Rearrange the figures in the number 51,937 to make the smallest possible number. . . . . . .
- *e.* Rearrange the figures in the number 436,852 to make the largest possible number. . . . . . .
- *f.* As compared with five-place numbers, are six-place numbers (always, or usually, or sometimes, or never) smaller? . . . . . .
- *g.* In the number 7,463 what is the largest possible number of hundreds? . . . . . . of thousands? . . . . . .
- *h.* Among the numbers 56, 93, 408, and 273 the one that has the largest possible number of tens is . . . . . ., and the one with the smallest possible number of tens is . . . . . .

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[14] The writer is grateful to the following persons, in addition to those mentioned in the text, for assistance in suggesting sample items: Mr. Lester Anderson, of the University of Minnesota, Dr. Arthur S. Otis, and Professor Leo J. Brueckner, Professor Ben A. Sueltz, and Professor Harry G. Wheat, the last three being members of this Committee.

i. 643 = 500 + ... 643 = 600 + 40 + ...
643 = 640 + ... 643 = 403 + ...
643 = 200 + 200 + ... 643 = 603 + ...

j. Using the figures 7, 0, 6, 3, 5 write a number with 0 in ten's place, 3 in one's place, 6 in hundred's place, 5 in ten-thousand's place, and 7 in thousand's place.

k. Write the four numbers that come next when you count by 1's:
28 ....
169 ....
396 ....
8857 ....
99998 ....

l. Suppose that:
✓ means one 1
○ means one 10
□ means one 100
: means one 1000

Using these symbols draw pictures for the numbers:
36 = ...
245 = ...
406 = ...
2310 = ...
3005 = ...

m. Write the four numbers that come next in counting
by 10's: 36 ....
by 100's: 857 ....
by 1000's: 7,007 ....
by 10,000's: 986,437 ....

n. Write the number that has:
6 tens and 3 ones .... 7 ones and 9 tens ....
54 hundreds and 7 ones ..... 73 tens ....
91 hundreds and 6 tens ..... 8 hundreds and 13 ones ...

o. Suppose that our number system had a basic unit of 8 instead of a basic unit of 10. Then 11 would mean 9 in *our* system (one 8 and 1), and 23 would mean 19 (two 8's and 3). Tell what the numbers below with a basic unit of 8 mean in *our* system with a ten base:
14 = ... 18 = ... 25 = ... 31 = ...

(2) *The Meaning of Whole Numbers*

a. Make 2 rows of circles, with 10 circles in each row. Cross out enough circles at the end of the second row to have 17 left.

b. Draw 10 balloons. Color 6 of them red.

c. Arrange the following numbers in size, from smallest to largest:
10, 23, 16, 47, 39, 310, 75, 469, 298

d. If you have 58 pennies and lay them on the table in rows of 10, you would have ..... rows of 10 each and ..... extra pennies.

e. Which number is about 4 times as large as 380?[15]

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[15] Items like those from (*e*) to (*h*) would need to be validated. Knowledge of the smaller numbers is most easily tested by means of actual objects, groups of which are constructed to match announced numbers. The larger numbers, if they

270 1500 2700 15,000

f. About how many 3¢-stamps could you paste side by side on a sheet of paper like this?
25 50 100 500

g. About how many pennies could you hold in both of your hands at the same time?
20 200 2000 20,000

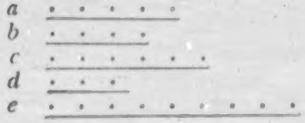
h. If you walked steadily for about an hour, about how many steps would you take?
100 10,000 100,000 1,000,000

i. In the number 9,037, the figure 9 represents how many times as much value as the figure 3?

(3) *The Meanings of Fractions, Decimals, and Per cents*

(The reader is referred to the excellent teaching and evaluation devices which are suggested by Miss Sauble on pages 157-195. In view of the number of examples there given, but a few supplementary ones are offered here. It should be clear that since common fractions, decimal fractions, and per cents are but three different ways of expressing parts or relationships between parts, the same device may frequently be used now to measure fraction meanings, at another time to measure decimal meanings, at still another time to measure per cent meanings.)

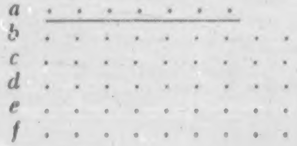
a. Study the lines below. Then answer the questions at the right.



b is ....% as long as *a*.
a is ....% as long as *b*
c is ....% as long as *a*.
a is ....% as long as *c*.
c is ....% as long as *d*.
d is ....% as long as *a*.
e is ....% as long as *a*.
e is ....% as long as *d*.

b. Blacken lines *b*, *c*, *d*, *e*, and *f* to make them as long in comparison with line *a* as you are told to make them.

Make *b* $\frac{5}{6}$ as long as *a*.
Make *c* $\frac{8}{6}$ as long as *a*.
Make *d* $\frac{2}{3}$ as long as *a*.
Make *e* $\frac{6}{6}$ as long as *a*.
Make *f* $\frac{4}{3}$ as long as *a*.



c. Study the number of dots in the following boxes and write your answers as decimals (per cents, fractions).

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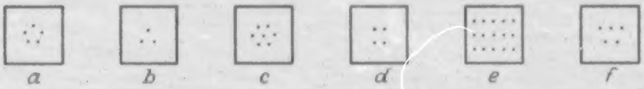
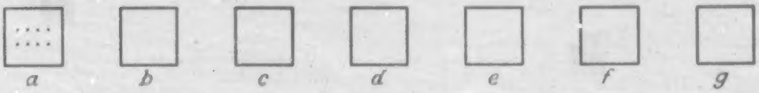
are abstract, are understood through computation, and hence testing involves more than knowledge of number size. If larger numbers are presented in concrete settings, as in (*f*), then lack of experience with some aspect of the setting may interfere with evaluation of number meanings. In all such items, too, the alternatives must be selected to suit the ages of the pupils tested, or better, their level of understanding.

Box *b* has ..... as many dots as box *a*.
Box *b* has ..... as many dots as box *d*.
Box *b* has ..... as many dots as box *e*.
Box *a* has ..... as many dots as box *b* (*c*, *d*, *e*).
Box *c* has ..... as many dots as box *a* (*d*).
Box *d* has ..... as many dots as box *a* (*b*, *e*).
Box *e* has ..... as many dots as box *b* (*a*).
Box *f* has ..... as many dots as box *a* (*b*).



d. Box *a* has 8 dots. Put as many dots in the other boxes as you are told to put in them.

In box *b* put $\frac{5}{8}$ as many dots as are in box *a*.
In box *c* put $\frac{3}{2}$ as many dots as are in box *a*.
In box *d* put 75% as many dots as are in box *a*.
In box *e* put $87\frac{1}{2}$% as many dots as are in box *a*.
In box *f* put 1.25 as many dots as are in box *a*.
In box *g* put .50 as many dots as are in box *a*.



e. Express the idea "3 out of 6" as a fraction .....; as a per cent .....

(4) *Important Technical Terms*

a. Study the examples at the right. Then copy a number that is:

$$\begin{array}{r} 463\\ 79\\ 5\\ +8743\\ \hline 9290 \end{array} \qquad \begin{array}{r} 464\\ \times 37\\ \hline 3248\\ 1392\\ \hline 27168 \end{array} \qquad \begin{array}{r} 96, \text{ R } 1\\ 6\overline{)577}\\ 54\\ \hline 37\\ 36\\ \hline 1 \end{array} \qquad \begin{array}{r} 563\\ -475\\ \hline 88 \end{array}$$

a multiplicand .....
a dividend .....
a subtraction remainder .....
a quotient .....
a multiplier ...
an addend .....
a partial product .....
a sum .....
a divisor .....

b. Copy the example in which you must borrow a thousand; a hundred; a ten.

$$\begin{array}{r} 805\\ -703\\ \hline \end{array} \qquad \begin{array}{r} 1846\\ -337\\ \hline \end{array} \qquad \begin{array}{r} 239\\ -48\\ \hline \end{array} \qquad \begin{array}{r} 4147\\ -936\\ \hline \end{array} \qquad \begin{array}{r} 7495\\ -6133\\ \hline \end{array}$$

c. Copy the example in which you must carry a ten; a thousand; a hundred.

$$\begin{array}{r} 346\\ +137\\ \hline \end{array} \qquad \begin{array}{r} 35\\ +42\\ \hline \end{array} \qquad \begin{array}{r} 4382\\ +810\\ \hline \end{array} \qquad \begin{array}{r} 14362\\ +3414\\ \hline \end{array} \qquad \begin{array}{r} 343\\ +285\\ \hline \end{array}$$

d. Copy from box A all the proper fractions; all the mixed numbers; all the improper fractions.

e. Copy from box B the fraction that has 6 for a numerator; 3 for a numerator.

f. Copy from box B the fraction that has 12 for a denominator; 2 for a denominator.

A	$5\frac{1}{6}$	$\frac{7}{4}$	$\frac{2}{3}$	$3\frac{6}{8}$
	$4\frac{5}{8}$	9	$2\frac{1}{3}$	

B	$\frac{2}{3}$	$\frac{12}{5}$	$\frac{6}{7}$	$\frac{8}{3}$
	$\frac{5}{6}$	$\frac{3}{8}$	$\frac{7}{12}$	$\frac{10}{2}$

g. Suppose you divided each of 3 apples into fourths, and you and your friends ate 7 pieces. The fraction showing how much was eaten would have the figure .... for the numerator and the figure .... for the denominator.

h. Five boys average 98 lb. in weight. The average (98) means (1) that all the boys weigh this much or more; (2) that none of the boys weigh this much; (3) that all the boys weigh exactly 98 lb.; (4) that all the boys would weigh 98 lb. if they weighed the same. Which? .....

(5) *Important Mathematical Generalizations*

a. Write the word *increased* or *larger*, or the word *decreased* or *smaller*, or the word *unchanged* or *same* in the blank space in each sentence. Write only *one* of these words.

If a number is multiplied by 0, that number is .....

When numbers other than 0 are added, the sum is ...... than any addend.

When whole numbers are divided by whole numbers other than 1, the quotient is ...... than the number divided.

Except when 0 is subtracted, differences or remainders are ...... than the numbers subtracted from.

If a number is multiplied by 1, that number is .....

In the division of whole numbers, divisors are usually ...... than the numbers divided.

If 0 is subtracted from a number, that number is ......

If a fraction or whole number is multiplied by fractions greater than 1, that fraction or whole number is ......

If you add the same number to both terms of a fraction, the value of the fraction is ......

b. To reduce a fraction to lower terms one must divide both numerator and denominator by ....

c. To change a fraction to a decimal you divide ...... by ......

d. Fractions cannot be added unless ......

e. Decimals can be changed to per cents by ....

f. Areas should always be expressed in terms of linear, square, cubic units. Which? ....

g. To divide 5.6 by 0.5 yields the same answer as to divide ..... by 5.0.

h. To multiply by per cents, one changes the per cents to ..... or to .....

(6) *Process Meanings*

a. Write M after the examples below in which the answers could be found by multiplying instead of by adding:

$4+4=$ .....
$7+7+7+7+7=$ .....
$3+5+7+9+2=$ .....

$75+46+98=$ .....
$49+49+49+49=$ .....
$18+108+76+23=$ .....

b. $4\times 9=36$. What other multiplication fact goes with this fact? What two division facts go with it?

c. $37\times 26=962$. Write three other relationships between these numbers which you know because $37\times 26=962$.

d. Write the idea of $24\div 6=4$ as a subtraction example.

e. How many seats has a room that has 5 rows of 14 seats each? Write first as an addition example; then as a multiplication example.

f. Write *add*, or *subtract*, or *multiply*, or *divide* in each statement.

To find the total of several unequal numbers, you .....

If you know how much one article costs and you know how many articles there are, you ...... in order to find how much they will all cost.

You know how much you had to start with and how much you have now. To find how much is gone, you ......

You want to give each of 6 boys an equal share of a number of marbles. To find how many each will get, you ......

To get the total of several numbers of the same size, the quickest way is to ......

You know how many sheets of paper you must have and how many you do have. To find how many more you must get, you ....

You know how much you spent and how many articles of the same kind you bought. To find how much each cost you ......

g. Draw dot pictures to show that the following answers are right or wrong. In subtraction you can cross out dots; in multiplication and division you can draw rings around groups of dots.

$12-8=3$
$16\div 4=4$
$3\times 6=21$

$5+6=13$
$24\div 3=9$
$15-8=7$

$7\times 4=28$
$17-9=8$
$12\div 6=3$

h. Write the number that completes each example:

10+6=8×......
23−7=4×......
63÷9=9−......
7−0= 8−......
8+7= 3×......
10−6=24÷......

i. Write the signs to complete the following examples:

7 ... 5=9 ... 3
36 ... 8=4 ... 7
49 ... 7=3 ... 4
4 ... 6= 3 ... 8
52 ... 7= 5 ... 9
72 ... 8=12 ... 3

(Simple verbal problems provide excellent test material for evaluating the understanding of the fundamental operations, as well as their significance or usefulness. Provided (1) that the computations are kept easy and (2) that unfamiliar terms and problem settings are avoided, success or failure in problem solving is pretty much confined to identifying the correct process to be used.)

(7) *Measurement as a Process, Plus the Meaning of Measures*

a. Write the word *more* or the word *fewer* in the blanks.

To change potatoes from peck measures to bushel measures you would need ..... measures.
You can get ..... dimes than nickels for a dollar.
You can get ..... 8-inch than 6-inch badges from a yard of ribbon.
If you change milk from gallon containers to pint bottles you will need ..... bottles than containers.
If you measure the length of a room with a foot ruler, you will need to lay it down ..... times than if you use a yardstick.
The larger the measure you use, the ..... times you use it in finding the amount of water in a barrel.

b. If one inch on a map means 25 mi., a distance of 150 mi. on the map would take ..... of space on the map.

c. Write the correct words in the blanks—*square inch, square foot, square yard, acre, square mile:*

The area of this classroom is about 1050 ......
The top of our automobile is 2½ ...... in area.
Mother used 64 ...... of cloth for a Christmas handkerchief.
The area of the Fair Grounds is 60 ......
My arithmetic book, when opened up, covers an area of 70 ......
Our country has an area of 6500 ......

d. Write the correct words in the blanks—*ounces, pounds,* or *tons:*

Our kitten weighs 15 ......
Fourteen men together weigh about 1 ......
My candy bar weighs 3 ......
A kitchen chair weighs about 12 ......
Six pencils together weigh about 4 ......
A large coal truck can carry 6 ......

(8) *Mathematical Relationships, Etc.*

a. None of the answers for any example is right, but one of the answers is nearer right than any of the others. Draw a ring around this number. Do not work the examples with paper and pencil.

In the first example you think, "88 is nearly 90; $7 \times 90 = 630$; 600 is nearest the answer, so I draw a ring around it."

In the second example, you think, "970 is nearly ..... tens; 98 tens divided by ..... is ..... tens; ..... is nearest the right answer, so I draw a ring around it."

In the third example, you think, "4960 is almost ..... hundreds, and 3120 is a little more than ..... hundreds; when I subtract I get ..... hundreds. So ..... is nearest the right answer, and I draw a ring around it."

Do the other examples in the same way.

$7 \times 88 =$	175	1500	98	(600)
$970 \div 8 =$	115	286	878	1750
$4960 - 3120 =$	7000	10,000	2000	700
$788 + 110 =$	650	900	1368	4600
$3042 \div 9 =$	27,500	350	750	110
$9 \times .89 =$	8	72	1.6	800
$13 \div \frac{7}{8} =$	112	15	43	200
$42\frac{2}{3} + 41\frac{7}{8} =$	140	90	10	380

b. Write the next two numbers in the incomplete series:

2	4	6	....	....	60%	52%	44%	....	....
$\frac{1}{9}$	$\frac{1}{3}$	1	....	....	.0006	.0048	.0384	....	....
89	67	45	....	....	108	36	12	....	....
4.5	1.5	0.5	....	....	12	6	3	....	....

c. By studying the first three examples worked out for you in each set below, see if you can tell what the missing number in the fourth example in each set should be. Do not work out the fourth example; just study the first three.

$18 \times 37 = 666$
$21 \times 37 = 777$
$24 \times 37 = 888$
$27 \times 37 = \ldots$

$1 \times 9 + 2 = 11$
$12 \times 9 + 3 = 111$
$123 \times 9 + 4 = 1111$
$\ldots \times 9 + 5 = 11{,}111$

$1 \times 8 + 1 = 9$
$12 \times 8 + 2 = 98$
$123 \times 8 + 3 = 987$
$1234 \times 8 + 4 = \ldots.$

$28 \times 15{,}873 = 444{,}444$
$35 \times 15{,}873 = 555{,}555$
$42 \times 15{,}873 = 666{,}666$
$.. \times 15{,}873 = 777{,}777$

$9 \times 9 + 7 = 88$
$98 \times 9 + 6 = 888$
$987 \times 9 + 5 = 8888$
$\ldots \times 9 + 4 = 88{,}888$

$1 \times 7 + 1 = 8$
$12 \times 7 + 2 = 86$
$123 \times 7 + 3 = 864$
$1234 \times 7 + 4 = \ldots.$

d. If the answers for the following problems are silly, write the correct answers.

A rooster standing on one leg weighs 6½ lb. When he stands on both legs he weighs 13 lb.

If it takes 3 min. to boil an egg, then it takes 18 min. to boil 6 eggs together.

If a man earns $5.00 a day, he earns $30.00 in 6 days.

To give each of his 5 friends a piece of pie and to keep a piece of the same size for himself, Henry cut a pie into 5 pieces.

Mary needed 18 in. of cloth to finish a dress, so she bought half a yard.

June who is very greedy took ⅝ of a stick of candy instead of ¾, so as to get more.

e. Dr. A. S. Otis has suggested the need for problems which can be solved by ingenious or resourceful procedures. One example is as follows:

How many badges 3½ in. long can be cut from a ribbon 22⅝ in. long, and how much ribbon would be left?

The usual solution is:

for the first part	for the second part
$22\frac{5}{8} \div 3\frac{1}{2} = \frac{181}{8} \div \frac{7}{2}$ $\frac{181}{\underset{4}{\cancel{8}}} \times \frac{\overset{1}{\cancel{2}}}{7} = \frac{181}{28} = 6\frac{13}{18}$ badges	$\frac{13}{28}$ of $3\frac{1}{2} = \frac{13}{\underset{4}{\cancel{28}}} \times \frac{\overset{1}{\cancel{7}}}{2} = \frac{13}{8}$ or $1\frac{5}{8}$ in.

The resourceful solution is:

for the first part	for the second part
Think that if one badge takes 3½ in., 2 of them will take 7 in. Then there will be as many *pairs* of badges as there are 7's in 22⅝ in.	
There are obviously 3 pairs (21 ÷ 7 = 3), or 6 badges, and	(without writing) there will be 1⅝ in. of ribbon left (22⅝ − 21 = 1⅝)

It is Otis' belief, which the writer shares, that this kind of resourcefulness can and should be taught.

(9) *Rationale of Computation*

(Some students of arithmetic would not teach all of these logical relationships to children. The writer would, but he offers the testing devices below with full awareness that many would not favor them. It should be understood that the writer here seeks to test understanding and not the thought processes actually to be used by children. These thought processes are of course much shorter and more direct.)

a. Study the work at the right. Then write the missing numbers or words.

$$\begin{array}{r} 2 \\ 7\overline{)169} \end{array}$$

The dividend is ...... It contains only 1 hundred, so that it cannot be divided by the divisor, which is ...... The dividend has ...... tens (+9 ones). 16 tens÷7=2 ...... The quotient figure is written above the 6 of the dividend to show that 2 means 2 ......

b. In the example $5\overline{)496}$ the first quotient figure is 9; it is written above the figure ...... in the dividend, to show it means 9 ......

c. In the example $3\overline{)862}$ the first quotient figure is ..... It is written above the figure ..... of the dividend, so as to show that it means so many .....

d. Study the example at the right. Then write the numbers or words that are called for.

$$\begin{array}{r} 4876 \\ +3417 \\ \hline \end{array}$$

6+7= .....; write ....., and carry 1 .....
Add the tens: 1+7+1= ..... (tens) Write 9 in ..... place.
Add the hundreds: .....+.....=..... (hundreds). Write ..... in the hundred's place, and carry 1 .....
Add the thousands: 1+.....+.....=8 ..... Write 8 in the ..... place.

e. In the subtraction example at the right, you cannot subtract the ones, 9 from 4, so you borrow 1 ..... from ..... 14−9= .....; write the figure in ..... place.

$$\begin{array}{r} 4364 \\ -2919 \\ \hline \end{array}$$

Subtract the tens: .....−1= ..... tens. Write the figure in ..... place.
Subtract the .....: 3−9=? You must borrow 1 ..... from ..... 13 (hundreds)−9 (hundreds)= .... Write the figure in ..... place.
Subtract the thousands: .....−2= ..... (thousands). Write the figure in ..... place.

f. In the multiplication example at the right the multiplier, ..., contains 3 ..... and ..... ones. When you multiply by 4, you multiply by 4 .....; when you multiply by 3 you multiply by 3 .....

$$\begin{array}{r} 12 \\ \times 34 \\ \hline 48 \\ 36 \\ \hline \end{array}$$

4×2=8 ..... Write 8 in ..... place, under 2 and 4.
4×1 (ten)=4 ..... Write 4 under 1 and 3 in ..... place.
3 (tens)×2=6 ..... Write 6 in ..... place, under 1 and 3.
3 (tens)×1 (ten)=3 ..... Write 3 in ..... place.

EXAMPLES OF OBSERVATION AND INTERVIEW

The few examples given below relate, for the most part, to computational difficulties. Nevertheless, they are described here under mathematical understandings, since the computational difficulties seem to have arisen from ignorance of principles, inadequate concepts, and the like.

Miss Thelma Tew,[16] in a fairly well-controlled experiment in which she undertook to teach children the division of fractions by the Common Denominator Method, noted two faulty procedures which were peculiar to this method. One procedure was to reverse the positions of dividend and divisor, as in solution (a). The other procedure was similar; it consisted in reversing the positions of the two numerators, the fractions, however, being retained in their correct positions. This procedure is illustrated in (b):

$$\text{(a)}\ \frac{2}{8} \div \frac{3}{4} = \frac{6}{8} \div \frac{2}{8} = 3$$

$$\text{(b)}\ \frac{2}{8} \div \frac{3}{4} = \frac{2}{8} \div \frac{6}{8} = 3$$

Both errors arose from the expectation of a whole number as the answer and the unwillingness to accept a fraction (1/3). The examples used to introduce the Common Denominator Method had all had whole numbers as answers. Once the nature of the error had been determined by observation of the pupils' written and oral work, the necessary remedial instruction was at once apparent.

Miss Hilda Briemson[17] interviewed children, using the Compass Survey Test for this purpose and reporting on their work habits with decimals, per cents, and common fractions. Four pupils arrived at the answer 1.75% for the example: $1\frac{3}{4}$ of N = ... % of N? Their explanation was, "Change the three-fourths to seventy-five hundredths. I knew that three-fourths is the same as seventy-five hundredths. One would be one hundred." The element of complete meaning is clearly lacking in this instance.

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[16] Thelma Tew, "Teaching Division of Fractions by the Common Denominator Method." Unpublished Master's thesis in education, Duke University, Durham, N. C. 1938.

[17] Hilda May Briemson, "Some Causes of Errors in Percentage." Unpublished Master's thesis in education, West Virginia University, Morgantown. 1936.

Miss Briemson, whose whole thesis is filled with excellent illustrations of the value of the interview, also found four pupils giving the answer 55/100% for the example: .55 of $N = \ldots\ldots\%$ of N? On being questioned they stated their procedure as, "I changed fifty-five hundredths, a decimal, to fifty-five one hundredths." Again, a very serious deficiency in meaning is revealed.[18]

Miss Edwina Deans, a second grade teacher in the Evanston, Ill., schools, has supplied in personal correspondence the following examples of observation and interview:

> "I asked one child how he knew $8+4$ so well. (He didn't have a good understanding of other combinations with sums of 11 and 12.) He said, 'I always remember the egg box you showed us.' We had used the egg box for finding the 4's and 3's in 12.
>
> "A couple of days ago Richard was reporting on some reading he had been doing. He said, 'A stage coach could go a thousand miles an hour!' It was interesting to hear the children's amazed comments. . . . 'The "400" (a Northwestern train) goes only 117 miles an hour. A train goes much faster than horses.' 'A horse can't run as fast as a car can go, and a train goes faster than most cars. A car goes only 55 or 60 miles an hour.' The conclusion was obvious: Richard was advised to look up his information again.
>
> "I wanted to see whether children who had been taught the meaning and composition of numbers could learn to borrow in subtraction. (This skill was not in the course of study; I was merely curious to see what such children could do with a minimum of instruction on this skill.) I wrote the example $62-37$ on the blackboard in vertical form. This is what Jimmy said, after Ruby had been 'stumped' in finding that she could not make the first subtraction $(2-7)$ at once: 'You could take 1 from 6.' I asked, 'Why can you do that?' Jimmy replied, 'Because it's 6 tens, and you need a ten over here, so it will make 12. 12 take away 7 is 5. (On questioning him later I found that he had gotten 5 by thinking '7 from 10 is 3, and 2 more is 5.') 3 from 5 is 2.'
>
> "When our radishes came up [this obviously was in connection with a garden unit or project] we had to decide how to think about them.

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[18] An excellent use of observation (analysis of written work) in this same area of percentage will be found in the reference: Arthur Edwards, "A Study of Errors in Percentage," *Twenty-ninth Yearbook of the National Society for the Study of Education, Part II,* Chapter XII. Public School Publishing Co., Bloomington, Ill. 1930.

Our problems were: (1) How many shall we leave in each row? (2) If we leave 4 in each row, shall we have enough for 2 radishes apiece? (We have 28 children in our room, and there were 10 rows of radishes.) (3) If we leave 5 in each row, shall we have enough for 2 radishes apiece? (4) What, if we leave 6 in each row?

"I had the children make dot pictures for counting by 4's, 5's, and 6's. No attempt was made to have the children learn the multiplication and division facts they discovered. I merely was trying to see how they could think in terms of these processes. Their work, as I observed them doing it, and also as I saw the results on their papers, gave me just the information I needed."

PUPIL REPORTS, PROJECTS, ETC.

No samples of evaluation by means of pupil reports, etc., are given here. The chief reason is that they consume a great deal of space. If the reader will refer to the discussion of this procedure on pages 240-241 preceding, he should be able to understand how number and quantity could appear in units, projects, and reports in ways which would reveal the presence or absence of mathematical understandings.

The writer has before him a second grade unit on the Post Office and the Mail, through which the teacher (Miss Edwina Deans) has been able to provide many experiences with number in circumstances which are as suitable for evaluation as for teaching. Such experiences are particularly helpful for disclosing the *level* of children's thinking. Thus, in comparing the cost of an ordinary 3¢ stamp and of an airmail stamp, the teacher can see whether they count, or use concrete objects (like marks or pennies), or can find the answer by the use of meaningful abstractions.

Sensitiveness to the quantitative in social relations, etc. The various outcomes within the third group of arithmetic outcomes (see page 232) lend themselves with differing degrees of success to measurement by means of tests. There are no standard tests to evaluate growth toward all these outcomes or toward any one fully. The nearest approaches are mentioned below. The first two outcomes in this group can probably be subjected to formal written testing better than can the others. Samples are given in the following pages.

For the evaluation of the other outcomes, the most fruitful procedures are (*a*) observation and (*b*) pupil reports, projects, and the like. With regard to observation, the teacher (1) may proceed informally, noting the extent to which pupils habitually describe personal uses of number in and outside the classroom, voluntarily bring matter from magazines and newspapers which contains quantitative data or accounts of business or engineering events in which number plays a large part, etc. Or, (2) he may arrange ahead of time situations which permit better controlled observations of children's use of number, if such is their habit. Perhaps no other classroom device offers more *opportunities* for either type of observation (if the opportunities are but grasped as they usually are not) than does the activity-unit, in which number occurs in a truly functional way.

As for pupil reports, projects, and the like, reference is to special pupil projects, reports, excursions, readings, dramatizations, picturizations, tabular presentations, billboard exhibits, scrapbooks, homemade models, etc., which reveal number in its historical and in its current social applications.

Neither (*a*) nor (*b*) has been exploited to anything like its possibilities. The close relation between teaching and evaluation is here clearly apparent. Children become sensitive to number and come to use number as they experience it in useful ways and are encouraged actually to use it; and these same conditions of use constitute the best basis for evaluation. Also, they help to insure that the habits of using number acquired in school will be available for use outside of school and later in adult life.

Varying purposes of evaluation. No value attaches to a discussion of the separate outcomes in terms of the purposes of evaluation (diagnosis, inventorying, etc.). While development in these areas of growth resembles growth in mathematical understanding in being gradual, about all that can be hoped for now is that teachers shall be aware of the need of providing opportunities for children to see the applications and the usefulness of number, that they shall actually provide them, and that they shall regularly observe the nature of their pupils' behavior in social situations involving number. Here again evaluation is subjective, but none the less valuable than if it were objective in the strictest sense.

TESTING DEVICES

Vocabulary, Knowledge of Business Practices, etc. Grossnickle[19] prepared a vocabulary test of sixty-eight business and social terms dealing with taxation, stocks and bonds, banking, insurance, merchandising, building and loan associations, and installment buying (only one item each for the last two). His test consists of multiple-choice items but has never been published or, to the writer's knowledge, standardized. The reference does, however, contain a useful list of important terms.

Part of the terms in the Vocabulary part of the Analytical Scales of Attainment (grades three and four, five and six, seven and eight) are such as fit in the present category—for example, exchange, withdraw, dues, overdue, wages, discount, cargo, profit, etc.

It is not particularly difficult to frame test items for such terms. (To make them good items is another matter.) Below in *a-h* are given samples in the multiple-choice and the Yes-No forms. In *i* is given another form of testing, which would seem to make the evaluation somewhat more functional.

a. The money one pays for protection on an insurance policy is called the (1) premium; (2) the dividend; (3) the discount; (4) the commission.

b. Or, the premium on an insurance policy is (1) the amount of the policy; (2) the money returned at the end of the year; (3) the length of the term of protection; (4) the payment made regularly for protection.

c. To receive money on a check made to our order we must (1) pay a tax on it; (2) indorse it; (3) discount it; (4) make out a note for it.

d. Or, when one indorses a check, one (1) writes his name on the back; (2) puts a date on it; (3) signs the note as maker; (4) deposits the money at the bank.

e. A mortgage on a property is one form of security of a loan. Yes No

f. Dividends from a company's earnings are paid on common stock before they are paid on preferred stock. Yes No

g. By the face of a note is meant the side of the note which contains the maker's name. Yes No

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[19] Foster E. Grossnickle, "Concepts in Social Arithmetic for the Eighth Grade," *Journal of Educational Research*, 30 : 475-488, March, 1937.

h. Customers receive a commission when they purchase goods of agents. Yes No

i. Write the correct word or words in each blank space. The words to be used are listed at the right.

The P and G Grocery Co. regularly takes its money to the bank every morning. Last Tuesday Mr. Black had this responsibility. Of course, he had to make out a .... showing all the cash and checks he was putting into the bank. Each check he had to .... for the company. The .... of one note was for $120.00. It was to draw .... at the rate of 5%, for 90 days, and the .... was October 1. The grocery company, however, wanted the money at once, so Mr. Black asked the bank to .... the note. This the bank was glad to do, and gave credit to the grocery company for the .... which amounted to $118.20.

interest
face
discount
deposit slip
date of maturity
indorse
proceeds

Graphs, Tabular Material, Statistics, etc. No samples are given under this heading, for the reason that many are to be found in textbooks and workbooks. It is rather easy to devise adequate test items of an objective character both in the way of constructing and in the way of interpreting statistical, graphical, and tabular data. In the interpretation of such data, pupils can be called upon, by means of multiple-choice, matching, and true-false techniques, to identify items, answer questions of fact, trace relationships, and the like.

Awareness of the Usefulness of Number, etc. The most valid evaluation is made by means of observation, pupil reports, projects, etc. An indirect means of measuring the outcomes under this heading is offered through objective test items which cover a great variety of social situations in which number is used. These situations for the purposes of evaluation must, of course, be selected with proper regard for the experiences normally to be expected of children of the given age. Samples are:

a. Is a city block often a mile long? Yes No
b. Do guns used in war sometimes shoot 10 mi.? Yes No
c. Would you expect to get 250 lollipops for 10¢? Yes No
d. Would 20 gal. of lemonade be needed for 30 children at a picnic? Yes No
e. Can you throw a baseball 500 ft.? Yes No

f. If you sleep 10 hr. a night, is that as much as 2 mo. a year? Yes No
g. Do we use seconds to express the length of a month? Yes No
h. Do we weigh coal as carefully as gold? Yes No
i. Are things sold in prepared packages in order to make them cheaper? Yes No
j. Do people hoard money when times are good more than when times are bad? Yes No Yes No
k. Are there eight sides on a cube? Yes No
l. Is term insurance the cheapest kind of life insurance? Yes No

OBSERVATION, INTERVIEW, PUPIL PROJECTS, REPORTS, ETC.

As has been stated repeatedly, formal tests have relatively little value for evaluating growth toward the objectives subsumed under quantitative sensitiveness. The reason is that even if testing devices can be constructed, they still are artificial. The essence of evaluation in this area is to "catch" the behavior in a functional setting, and tests are not functional so far as the uses, appreciations, etc., in this group of outcomes are concerned.

On the other hand, observation and the use of pupil projects, reports, and the like can be most effective for evaluation. The reader is again referred to Miss Sauble's chapter, especially pages 181-195, where are presented samples of units and projects which contain, or may be made to contain, many opportunities for evaluation.

CONCLUDING STATEMENT

The concep[illegible] of evaluation which has been advanced in the foregoing pages obviously places a heavy responsibility upon the classroom teacher. Standard tests, when properly selected and properly used, are of some service in meeting this responsibility. More helpful, when evaluation is undertaken for the direction of learning, are locally prepared paper-and-pencil tests. Nevertheless, throughout this discussion far the greater emphasis has been placed upon the insights which are to be had from the continuous and enlightened questioning and observation of children while they are engaged with their daily work in arithmetic and with their projects, units, and the like, in which arithmetic plays an important part.

If it be granted that the purposes and means of evaluation here outlined are valid, we face the task of implementing this conception. Certain obstacles which at present hinder progress must be eliminated. In the first place, teachers of arithmetic must receive a different kind of training from that to which they are usually subjected. Their training must be made functional. This is but another way of saying that they must be prepared for their teaching duties in a way which reveals to them the peculiar nature of those duties. Among other things, they must examine the subject matter they are to teach in the light of modern theory, which stresses, besides computational efficiency, both the mathematics and the social applications of arithmetic. Too, they must view the subject matter as the learner views it, and not merely as does the adult who has completed his learning. Otherwise, teachers can hardly appraise correctly and sympathetically the difficulties children meet in developing skill in quantitative thinking. The child-view of arithmetic is best attained by working directly with individual children and by interpreting what is observed in terms of growth toward this ultimate objective. Even so, programs of teacher training (and so also of classroom teaching and evaluation) will be handicapped until the second and third obstacles to progress have been removed.

The second obstacle is our ignorance of the characteristics and nature of sound arithmetical learning. As is pointed out elsewhere in this volume, little research relating to arithmetic has been oriented with respect to the child and his problems in learning. Compared with the large amount of research on other phases of arithmetic, the amount devoted to careful investigations of learning as such has been small indeed. This general neglect of the child and his learning is reflected also in the measures which have been used for evaluating teaching methods and devices, for example. Seldom indeed does one encounter evidence that research workers have been concerned with the qualitative or subjective phases of learning. If teachers are to have the information they sorely need to improve their instruction and their evaluation, research must be given a new direction, a direction which is well indicated at the end of Professor McConnell's chapter.

The third obstacle which prevents the implementation of evaluation of the kind here advocated resides in current materials of instructions. Many of our arithmetic textbooks are still little more than outlines of the subject in terms of the skills to be taught, and most workbooks still deserve the designation of drill pads and practice booklets. In textbooks and workbooks alike, explanations of operations and computational activities are commonly restricted to directions of what to do and how to do it. The underlying mathematical principles are not consistently called to the attention of pupils (or of teachers). Again with the exception of verbal problems, which after all contain few social applications that represent vital and real needs on the part of children, textbooks and workbooks do relatively little to instill an awareness of the significance of arithmetic and to develop habits of using arithmetic. And still again, textbooks and workbooks are commonly almost silent on suggestions for evaluating the degree to which children understand what they learn and are sensitive to the quantitative aspects of their lives. The improvement of texts and workbooks is one of the surest and most immediate means of redirecting teaching and evaluation in the way described in the foregoing pages.

None of the three obstacles here discussed is insurmountable. Likewise, none of them will be removed merely by pious hopes. If these obstacles are to be eliminated, they will be eliminated only by diligent and enlightened effort.

Chapter XI

RECENT TRENDS IN LEARNING THEORY

Their Application to the Psychology of Arithmetic

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LEARNING IS A CHANGE IN THE ORGANIZATION OF BEHAVIOR

Organization a primary aspect of behavior. Newer trends in the psychology of learning emphasize the primacy of organization. This principle can be illustrated in a variety of ways. For example, human beings—and animals for that matter—respond to *relations* among stimulus objects. This characteristic of behavior was clearly shown in a series of experiments on "equivalent stimulation," designed to permit variation in details of the stimulus situations to which the subjects reacted, while the general pattern in the situations was held constant. Among the several experimental tasks were (1) a difficult number maze in which all the numbers on the sheet were changed by certain sums in the several variations, and (2) a complex pencil maze in which variations were secured by printing the maze in different scales. The experiments demonstrated, first of all, that when stimuli were varied within certain rather wide limits as far as their absolute characteristics were concerned, learning did not suffer provided the internal organization of the stimuli remained constant. The results indicated, furthermore, that what the subjects learned in these experiments was not a loosely connected series of specific reactions but a highly organized response. In the pencil maze task, for example, the group which practiced with a maze of the same pattern but of different size on each repetition learned as efficiently as the group which practiced with mazes identical in both size and pattern. The former group could not have practiced the same movements each time. They learned, not a chain of specific responses, but an organized reaction pattern in which the particular movements were subordinate to the general configuration.[1] Such data as the results of these experiments emphasize the principle that learning is not the acquisition of items of information or skill, or of a multitude of discrete reactions, but is a *change in the organization of behavior* which gives the individual more effective control over the conditions of experience.

Form or plan facilitates memorization. Research on human learning has also shown that the presence of *form* or *plan* in material to be learned greatly facilitates memorization if one discerns the pattern. Thus, a list of numbers arranged according to a definite scheme can be learned much more easily than an "unformed" series of the same length.[2] Recent studies have revealed that some kinds of grouping, or organization, are more effective than others in learning. In one of the experiments, four comparable groups of subjects were required to learn the following series of numbers in four different ways.

2 9 3 3 3 6 4 0 4 3 4 7
5 8 1 2 1 5 1 9 2 2 2 6

Group I was told that the numbers were arranged according to a principle, and that both rows were built according to the same rule. For Group II the numbers were presented in the following way: 293 336 404 347. To Group III the numbers were given as amounts of government expenditures, that is, as $2,933,364,043.47 in 1929 and $5,812,151,922.26 in 1936. Group IV was given a lecture on government expenditures in which the following numbers were presented and referred to with the proper experimental frequency.

$2,933 million $15,192,226,000
$5,812 million $36,404,347,000

The results greatly favored the first of the four types of arrangement on both immediate and delayed reproduction tests. The investigator concluded that various types of grouping may aid learning, but that some forms of organization are more "adequate" than others. The most adequate, he concluded, were those based upon intrinsic relations. The organization of a series of numbers according to a principle is an example of the kind of grouping which seems most effective for learning and delayed recall.[3]

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[1] R. Leeper and D. O. Leeper, "An Experimental Study of Equivalent Stimulation in Human Learning," *Journal of General Psychology*, 6 : 344-376, April, 1932.

[2] J. P. Guilford, "The Role of Form in Learning," *Journal of Experimental Psychology*, 10 : 415-432, October, 1927.

Intrinsic relations in number system. In arithmetic, the number system provides the intrinsic relations which constitute the basis for understanding and organizing the multitude of specific skills and abilities which are included in it and controlled by it. This closely knit system of ideas, principles, and processes has a meaning which will not be revealed by dealing with the elements alone. By failing to teach the basic principles of the decimal system, and by requiring the pupil merely to memorize a host of discrete number facts, we deprive him of the only effective means of generalizing his number experiences and of applying his learning intelligently in new situations. Unfortunately, arithmetic has been analyzed, as Wheat explains, "into a multitude of combinations, processes, formulas, rules, types of problems, etc., and the pupil is taught each in turn as a separate item of experience. Often, when he has completed the course, he knows only those parts that he can still remember, and they all seem to him as separate and unrelated combinations, processes, formulas, rules, and types of problems to be solved. Finally, when his memory for these separate items fails him, he has nothing left to carry into his adult world but the meaningless, uninteresting, and unpleasant experiences that his classes in arithmetic seem to have provided for him."[4] The purpose of systematic learning in arithmetic, on the other hand, is to provide pupils "with methods of thinking, with ideas of procedure, with meanings inherent in number relations, with general principles of combination and arrangement, in order that the quantitative situations of life may be handled intelligently. . . ."[5]

Learning as differentiation. The principle that organization is a primary aspect of human behavior is also illustrated by the fact that the most elementary adjustments involve highly structured activity. The first efforts of the child to manage the quantitative aspects of experience provide evidence that although his behavior is relatively *undifferentiated,* or unparticularized, it is none the less organized. The course of development in behavior is often from the whole to the part, from the general to the specific. This process of growth is called differentiation, which has been defined as "the emergence of a feature or detail of the original pattern out of its setting to become a new and particularized whole."[6] The phenomenon of the differentiation of specific details from a more generalized response is evident in the way in which the child learns to count. Contrary to the usual assumptions, the child's number ideas do not begin with counting. A recent study indicates, on the contrary, that "a group of objects meaning 'many' to the child is not differentiated first into a multitude of ones. A gross differentiation precedes this more complex differentiation; the division of the group is made into more or less unequal subgroups. . . ."[7] The child does not need to count to comprehend the meaning of more or less. The idea of equal is somewhat more difficult, but can also be grasped without counting. The study just quoted revealed that it was only at a later stage that a group was divided into homogeneous parts on the basis of number. The data also showed that the cardinal and ordinal ideas of number emerge together. Counting accompanies this process as a relatively *late* phase of differentiation. Counting is a particularized, specific, refined method of dealing with groups of objects. But it is also important to note that counting provides a systematic, organized means of dealing with the discriminated individual members of a group. It involves a perception of the relationship of the part to the whole.[8] Counting is a means of discriminating the members of a group. It is also the means of grouping or organizing individual items.

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[3] G. Katona, *Organizing and Memorizing*. Columbia University Press, New York. 1940.

[4] H. G. Wheat, *The Psychology and Teaching of Arithmetic*, p. 157. D. C. Heath and Company, Boston. 1937.

[5] *Ibid.*, p. 140.

Differentiation has also been defined as "the progressive explication of detail in an implicitly apprehended whole." Beginning with a relatively undifferentiated pattern, the structure or the significance of the whole may become more explicit and the details which are relevant may become more distinguishable. This continuous differentiation of an idea or process occurs constantly in meaningful learning in arithmetic. Wheat points out that the idea of ten and the idea of position are two of the core ideas which run throughout arithmetic, systematically taught and learned, from beginning to end. He explains the development of these ideas as follows:

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[6] W. D. Commins, *Principles of Educational Psychology*, p. 334. Ronald Press Co. 1937.

[7] N. M. Russell, "Arithmetical Concepts of Children," *Journal of Educational Research*, 29 : 647-663, May, 1936.

[8] Wheat, *op. cit.*, p. 20.

> Even a glimmering of these ideas at the beginning helps the pupil to recognize and rely upon them in the adding and subtracting of two-place numbers. "Carrying" tens in addition, subtraction, and multiplication, and division by two-place numbers gives opportunity for employing the ideas. At every step, to and throughout decimals and percentage, these same ideas continue to appear. Every step may provide the occasion both of illustrating and extending these ideas in the pupil's mind, and of giving him an opportunity to use what little he may previously have learned of them as an aid in his attack upon new processes to be learned. . . .
>
> Related to the ideas of ten and position—really a part of them—are the ideas of *size* and of *number* (whether of parts or of groups). Although these may be most evident in fractions, they may be understood the better if one can discover them running through the whole of arithmetic. And fundamental to the whole of arithmetic are the ideas of combination of unequal and equal groups in addition and multiplication and of separation into unequal and equal groups in subtraction and division respectively.[9]

These comments are excellent illustrations of the way in which ideas incompletely understood in the beginning acquire more significance through experience and ultimately embrace a much more extensive set of particulars and applications. They also illustrate the principle that however imperfect or vague the pattern and the details may be in the beginning, however well rounded out the pattern, and however specific and distinguishable the details may become as learning goes on, unity characterizes the learning process throughout.

Learning as integration. Learning, of course, is more than a process of differentiation. It is one of integration and reorganization as well. Integration occurs when one discovers the relations between things which were learned at different times and in different contexts. Mechanistic theories of learning of necessity have had to make provision for some kind of association of "unit skills" and "lesser abilities" into an operating hierarchy. But the assembling of parts was primarily in terms of relatively self-contained or discrete processes such as addition of whole numbers, long division, and Case I of percentage. This form of integration was little more than summing, resulting ordinarily in a series of procedures conducted by rule-of-thumb methods. Understanding the mathematics, or logic, of a process, and relating one process to another and to basic number ideas, was not only not recommended but was very frequently discouraged.

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[9] *Ibid.*, pp. 147-148.

Integration, rightly conceived, involves a highly coherent organization. It is essentially a meaningful rather than a mechanical process. This connotation of "integration" is clearly implicit in the following definition of the term:

> . . . whenever a number of more or less discrete objects or "ideas" enter into a configuration of behavior, they become joined by virtue of their membership in the whole; the members are thereafter held together, not by the external agency of an associative "glue," but by the transformation they have undergone in losing something of their individuality and becoming the members of a single pattern of behavior.[10]

There are many organizations of lesser or greater degree in arithmetical processes. For example, one may think of a systematic arrangement of the addition facts or of the subtraction facts. One may also consider the relationships between addition and subtraction, not only of specific "facts," but of the processes themselves. The four fundamental processes—addition, subtraction, multiplication, and division—may be conceived as different but related ways of regrouping, involving combination and separation of unequal and equal sub-groups. Regrouping by tens is another means of utilizing relationships. As learning proceeds, one should construct more inclusive and systematic organizations of ideas and processes governed by the fundamental structure of the number system. Thus it is not only desirable to see the correlation of the "three cases" in percentage, and a generalization

[10] R. M. Ogden and F. S. Freeman, *Psychology and Education*, p. 179. Harcourt, Brace and Company, New York. 1932.

of the underlying number relationships, but also to understand the relations between fractions, decimals, and per cent. As the pupil studies the applications of percentage, such as interest, savings, investments, gain and loss, cost and selling price, insurance, taxes, etc., he should not only secure training in practical activities but also a better understanding of percentage. "Such a procedure," writes Wheat, "brings into a single, unified scheme of thinking or method of attack what otherwise might easily be a dozen separate, distinct, and unrelated 'applications' of percentage."[11] The process of integration, or reorganization of experience in more mature, effective, and systematic form, is but another recognition of the movement to make relatedness and organization the central concept in the psychology of learning.

The mechanization of arithmetic. "On the one hand," says Wheat, "number as a science is systematic and consistent; on the other hand, number as a practical art is often a series of rule-of-thumb procedures."[12] This observation is only too true. Why has instruction in arithmetic disregarded and obscured the inherently sensible and understandable structure of the subject?

One of the principal reasons for the mechanization of arithmetic is that it has been caught in the toils of the connectionist theory of learning. Mathematics, particularly arithmetic, has been easy prey for analysis into elements, bonds, connections. The most influential person in the movement to psychologize arithmetic in conformity with connectionist principles, has, of course, been Thorndike. Under his leadership, psychologists have prepared the arithmetic for learning by analyzing it into hosts of specific items. This followed naturally from the habit of breaking behavior into small units, and attempting to describe the most complex processes by listing their constituent parts. The detailed items into which subject matter could be dissected corresponded nicely to the specific S—R connections of which any mental function supposedly was composed. These psychologists conceived of learning as the process of forming specific bonds and gradually seriating or combining them until the function as a whole had been constructed. Correspondingly, one memorized the items of subject matter piecemeal and ultimately collected them into more or less compact hierarchies best described in many instances as a connected series of routine procedures. Under this kind of learning, there was no need to establish a scheme, a fundamental idea, or a basic pattern of responses when beginning to acquire a skill or ability. Neither was there any assurance that generalized control of the process would emerge in the end.

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[11] Wheat, *op. cit.*, p. 150.

[12] *Ibid.*, p. 153.

Much has been said in discussions both of learning and curriculum construction about the differences between the psychology and the logic of a school subject. In many instances, there is far more relation between the two than superficial considerations would suggest. In the case of arithmetic, attempts to psychologize the subject appear to have damaged it both logically and psychologically. By decomposing it into a multitude of relatively unrelated connections or facts, psychologists have mutilated it mathematically, and, at the same time, by emphasizing or encouraging discreteness and specificity rather than relatedness and generalization, they have distorted it psychologically. They have obscured the systematic character of the subject, and have created a doubtful conception of how children learn it. Furthermore, the practice of connectionism in arithmetic leads almost inevitably to immediate emphasis on rapid and accurate computation rather than on the development of the ability to think quantitatively. Computation is easily segmented into constituent units, while quantitative thinking, because it rests upon generalized understanding, is not susceptible to analysis into specific elements.

A recent writer calls the meticulous analysis of arithmetical processes into detailed elements, or even into type examples, "academic insanity." That seems a little over-severe. However one learns arithmetic, he must ultimately be able to respond correctly and efficiently to a considerable number of variants of any process. Inventories of the characteristic forms which a process may take are useful in the construction of diagnostic examinations and in the preparation of varied experiences with which to attain basic understandings and procedures. However, one can no longer defend the learning of a large number of relatively independent items within a process as a substitute for understanding the process mathematically. Evidence is accumulating slowly but surely which reveals that when the learner understands the number system and the operations which its structure permits, he has developed insight into arithmetical processes which makes instruction and drill on each variant or every specific "fact" unnecessary.

Social utility related to excessive analysis. The reduction of arithmetic to the specific abilities which have social utility has played handmaiden to the connectionist theory of learning. As Wheat has pointed out, when social usefulness determines entirely what arithmetic one should learn, ". . . any idea of a relation between topics and processes must be abandoned. When topics and processes may be included or excluded at will, any relationship among them must be decidedly unimportant."[13] However, if one is as seriously concerned with understanding arithmetical processes as with using them in practical situations (this writer believes that the one is in fact the most effective means of achieving the other), he will include in the learning sequence whatever is necessary to give mathematical significance and logical structure even if some of it has no *immediate* social utility. The science of number provides a rationale for the meaningful organization of experience, a scheme which the learner can use to systematize his ideas and his computational skills. This point of view has little affinity for connectionist dogma with its emphasis on blind repetition and its wariness of rational procedures. On the contrary, it is part and parcel of a theory of learning which stresses organization rather than discreteness, understanding rather than memorization, the exercise of the higher mental processes rather than dependence upon lower-order habits. "The psychology of the higher mental processes," Judd insists, "teaches that the end and goal of all education is the development of systems of ideas which can be carried over from the situations in which they were acquired to other situations. Systems of general ideas illuminate and clarify human experiences by raising them to the level of abstract, generalized, conceptual understanding."[14]

[13] *Ibid.*, p. 125.

[14] C. H. Judd, *Education as Cultivation of the Higher Mental Processes*, p. 177. The Macmillan Company, New York. 1936.

LEARNING IS A DEVELOPMENTAL PROCESS

Confusion of product and process of learning. That learning is a process of development is a second principle of far-reaching importance for education. Association theories confused the *end* or the product of learning with the process; they treated learning as the fixation of responses. In spite of all the revisions which Thorndike has made in his statement of the principles of learning, his interpretation of the business of teaching simmers down to this: Identify the situation for the learner; identify the response, and make sure that the learner can make it; then put him repeatedly through the reaction and make the practice satisfying. Connectionism comes perilously close to treating "situations," "responses," and "bonds" as entities in themselves, as items which have independent existence. The literalness with which some associationists speak of these elements is little short of the idea of cords of different degrees of strength actually tied at either end to definite objects.

To the connectionist, the only differences between responses at the inception and at the completion, or fixation, of a reaction pattern are the speed of reaction and the variability of the response. In fact, the bond psychologist looks with suspicion upon any attempt to invest a task with meaning, or to approach the final stage of habituation through intermediate or developmental reactions. Thorndike insisted, for example, that "time spent in understanding facts and thinking about them is almost always saved doubly by the greater ease of memorizing them." Again, he declared:

> The doctrine that the customary deductive explanation of why we invert and multiply, or place the partial products as we do before adding, may be allowed to be forgotten once the actual habits are in working order, has a suspicious source. It arose to meet the criticism that so much time and effort were required to keep these deductive explanations in memory. . . . The fact was that the pupil learned to compute correctly irrespective of the deductive explanations. They were only an added burden. His inductive learning that the procedure gave the right answer really taught him. So he wisely shuffled off the extra burden of facts about the consequences of the nature of a fraction or the place values of our decimal notation. The bonds weakened because they were not used. They were not used because

> they were not useful in the shape and at the time that they were formed, or because the pupil was unable to understand the explanations so as to form them at all.
>
> The criticism was valid and should have been met in part by replacing the deductive explanations by inductive verifications, and in part by using the deductive reasoning as a check after the process itself is mastered. . . . What is learned (i.e., the deductive theory of arithmetic) should be learned much later than now, as a synthesis and rationale of habits, not as their creator.[15]

In another instance, Thorndike said that "if an arithmetic process seems to require accessory bonds which are to be forgotten, once the procedure is mastered, we should be suspicious of the value of the procedure itself."[16]

The same attitude toward arithmetic seems to be behind an unfortunate statement in the recent yearbook on *Child Development and the Curriculum* which divided the school curriculum into two broad classes of material, the one involving processes of development and reorganization, the other including "skills and knowledge that are acquired through *specific practice*, such as reading, *arithmetical computation*, playing the piano, the facts of history, and so forth. . . ."[17] On the contrary, there is now evidence that even skill in arithmetical computation involves a process of growth, and that the purpose of teaching is to give constructive guidance to this course of development. This view, instead of confusing process and product, emphasizes the progressive changes in the learner's behavior, and recognizes that each stage in the development of behavior is an outgrowth of a previous one.

Developmental studies in arithmetic. There are at least three studies which have demonstrated the distinction between *process* and *product* in the learning of arithmetic. These investigations have shown that the acquisition of arithmetical abilities involves developmental sequences and continuous reorganization of behavior in which more mature forms of response are substituted

[15] E. L. Thorndike, *The Psychology of Arithmetic*, pp. 115-116. The Macmillan Company, New York. 1925.

[16] *Ibid.*, p. 114.

[17] J. E. Anderson, "Problems of Method in Maturity and Curricular Studies," *Child Development and the Curriculum, Twenty-eighth Yearbook of the National Society for the Study of Education, Part I*, Chapter XX, pp. 400, 401. Public School Publishing Company, Bloomington, Ill. 1929.

for less mature but nevertheless essential or useful steps in understanding and skill.[18] These studies indicate clearly the following significant facts:

(1) Abstract ideas of number develop out of a great amount of concrete, meaningful experience; mature apprehension of number relationships can be attained in no other way. Furthermore, the adequate development of number ideas calls for systematic teaching and learning.

(2) Drill does not guarantee that children will be able immediately to recall combinations as such.

(3) Habituation of number combinations is a final stage in learning which is preceded by progressively more mature ways of handling number relationships.

(4) Repeating the final form of a response from the very beginning may actually encourage the habituation of immature procedures and seriously impede necessary growth.

(5) Drill as such makes little if any contribution to growth in quantitative thinking by supplying maturer ways of dealing with number.

(6) Intermediate steps, such as the use of the "crutch" in subtraction, aid the learner both to understand the process and to compute accurately. With proper guidance, these temporary reactions may be expected to give way to more direct responses in the later stages of learning.

(7) Reorganization of behavior occurs as the child's understanding grows, and results in the emergence of more precise, complex, and economical patterns of behavior.

(8) Understanding the number system and the methods of operation it makes possible facilitates both quantitative thinking and, ultimately, rapid and accurate computation.

Most of the investigations of methods of instruction in arithmetic have measured only computational abilities. When we extend the measurement program to evaluate instruction in terms of outcomes in quantitative thinking, the importance of meaningful and developmental activities will probably prove even greater.

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[18] W. A. Brownell, *The Development of Children's Number Ideas in the Primary Grades.* University of Chicago Press, Chicago. 1928.

W. A. Brownell, and C. B. Chazal, "The Effects of Premature Drill in Third-Grade Arithmetic," *Journal of Educational Research,* 29 : 17-28, September, 1935.

W. A. Brownell, *Learning as Reorganization.* Duke University Press, Durham, N. C. 1939.

LEARNING IS A MEANINGFUL PROCESS

The discussion of the primacy of organization and the developmental character of learning has already indicated, by implication at least, the third principle of learning for which there is now a substantial body of evidence: *Learning is a meaningful rather than a mechanical process.* Fundamentally, meaning inheres in relationships. Relationships are established by the control which some organization or system exercises over the parts. The "meaning theory" of arithmetic instruction, therefore, emphasizes the importance of teaching children to understand our decimal number system and the ways of manipulating it.[19] This system provides the basic pattern for understanding and relating the many specific items which are included in it and controlled by it.

Those who really believe that learning is a meaningful process, instead of merely making a verbal concession to what they suspect is a passing fad, will insist that a general halo of meaning is not enough. There is in certain quarters a lingering desire to present a few of the number combinations meaningfully, and then, in this atmosphere of reasonableness, to resort to formal drill for the remainder of the combinations. On the contrary, learning should be meaningful throughout, not merely during the first few steps. Organization, meaning, and development are essential characteristics of the learning process, not incidents, nor elective devices. It is surprising how fearful of meaning some persons are. It has been suggested, for example, that although it might be defensible to use the subtraction "crutch" in order to explain the process, it should then be immediately discarded for drill in the most mature form. This proposal, however, is based upon the assumption that "understanding" and "fixation" are distinctly separable functions, a presupposition which the theory of learning as development cannot sustain.

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[19] W. A. Brownell, "Psychological Considerations in the Learning and the Teaching of Arithmetic," *The Teaching of Arithmetic*, pp. 1-31. Tenth Yearbook of the National Council of Teachers of Mathematics. 1935.

Distinction between social and meaningful arithmetic. There has been a tendency to assume that learning arithmetic in social situations and for social purposes makes it *meaningful* arithmetic. But a moment's reflection will lead to the realization that specific training in the employment of an arithmetical procedure in a social situation may make no contribution whatever to the understanding of that process as such. The mere fact that the words "marbles" or "pennies" are attached to formal repetition of 9+8, for example, may have little to do with the child's insight into the number relation behind the verbalization. It is important, of course, to learn the arithmetic which is useful in daily life, and to apply arithmetical processes in as many social situations as possible. Using these operations as social tools undoubtedly invests them with meaning in a very restricted sense. Fundamentally, however, to learn arithmetic meaningfully it is necessary to understand it systematically.

Buckingham has supplied a terminology to cover the social phases of arithmetic and to restrict the use of "meaning" to its mathematical reference. For the social implications of arithmetic, he proposes the term "significance." By the significance of number he means "its value, its importance, its necessity in the modern social order . . . the role it has played in science, the instrument it has proved to be in ordering the life and environment of man." He continues:

> Under the head of meaning I include, of course, the rationale of our number system. The teacher who emphasizes the social aspects of arithmetic may say that she is giving meaning to numbers. I prefer to say that she is giving them significance. In my view, the only way to give numbers meaning is to treat them mathematically. . . . I hasten to say, however, that each idea supports the other. . . . The one emphasis will exalt arithmetic as a great and beneficent human institution, the supporter of a fine humanistic tradition. The other emphasis, the mathematical one, will lift arithmetic, even in the primary grades, from a formalized symbolism to the dignity of a quantitative system. In short, when arithmetic is taught with meaning, it ceases to be a bag of tricks and becomes, as it should be, a recognized branch of mathematics. . . .
>
> When we confront children with a significant and meaningful experience, when they make the experience theirs, they acquire insight, each to the degree that he is able. More specifically, and in particular relation to number, they gain in two ways: they understand number

as such, and they also understand when and how to use it to serve their purposes. . . .[20]

The problem of curriculum organization. The problem of meaningful learning in arithmetic, as Buckingham defines it, has extremely important bearing upon the nature of curriculum organization. Shall arithmetic be taught as a systematic subject, or should the pupil acquire arithmetical abilities incidentally, i.e., in connection with other subjects, or only as they become a part of purposeful life activities? The issue in arithmetic, of course, is only one aspect of the broader problem. This problem is often stated as follows: Should the curriculum be composed of systematic bodies of subject matter, logically organized, or should it be composed of life experiences or activities in which subject matter is utilized and organized without respect to conventional academic sequences? The latter alternative may be illustrated by Harap's study, in which pupils learned decimals by using the processes in practical activities which made the computations necessary.[21] In appraising the outcomes of learning in arithmetic under this scheme, the ability of the pupil to make the necessary computations in the particular situations in which practice is secured is not the most important outcome. The essential abilities are those which enable the individual to discern the quantitative aspects of a great variety of specific situations, to choose the proper procedures in the light of this understanding, and to perform the relevant operations efficiently. There is very good reason to believe that this kind of quantitative thinking depends upon a mathematical understanding of arithmetic, upon a systematic study of the science of number. The issues stated above were couched in the usual "either-or" dichotomy. Actually, we should not be choosing between mutually exclusive alternatives. We should cultivate two fundamentally important types of relationships. We should have a wealth of experience, on the one hand, in bringing understandings, information, abilities, and skills from many sources—from many subjects, so to speak—to bear in unified fashion upon significant problems of adjustment. We need, on the other hand, to organize our learnings in the highly systematic form which fields and subjects of knowledge make available. Dewey has pointed out that "to grasp the meaning of a thing, an event, or a situation, is to *see it in its relations to other things;* to note how it operates or functions, what consequences follow upon it, what causes it, what uses it can be put to."[22] To use a mathematical process in proper coordination with other factors in the solution of a practical problem gives it significance; to see it in relation to other processes in its *own* highly organized system endows it with meaning. Broadly speaking, generalized ideas with which to interpret and control new situations are the means of continuous adjustment, or of the utilization of previous experience under different conditions. To acquire these generalizations meaningfully, it is very often necessary to learn them in the context of the closely integrated logical system to which they belong. Arithmetic is no exception. To disregard the mathematical relations of number processes means a reversion to a form of teaching, the outcomes of which were described several years ago as follows:

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[20] B. R. Buckingham, "Significance, Meaning, Insight—These Three," *The Mathematics Teacher*, 31 : 24-30, January, 1938.

[21] H. L. Harap and E. Mapes, "The Learning of Decimals in an Arithmetic Activity Program," *Journal of Educational Research*, 29 : 686-693, May, 1936.

> There are many pupils in school who succeed in learning their arithmetic only as a mass of isolated and unrelated number facts. . . . Through constant drill and persistent effort, they succeed in acquiring skill in the operations, but fail to recognize withal the nature and the meaning of the operations. As a result they learn to add, subtract, multiply, and divide with a fair degree of mechanical precision; they learn to perform such operations as they may be directed to perform; but they do not develop the ability to recognize the presence of these operations in the simplest practical situations in which they may be found. In the course of time, however, they succeed in remembering that a statement which includes such terms as "how many," "altogether," "total," "sum," etc., requires addition. . . . In other words they learn to remember the various computations; they come to regard them as so many mechanical and meaningless performances; and, finally, they learn such of their applications as can be remembered by formula and rule.[23]

Using arithmetic in practical affairs and learning it as a system of thought are both essential to the development of the ability to think quantitatively.

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[22] John Dewey, *How We Think*, p. 137. D. C. Heath and Company, Boston. 1933.

[23] H. G. Wheat, *The Psychology of the Elementary School*, pp. 142-143. Silver Burdett Company, New York. 1931.

Learning is thinking. Meaningful learning emphasizes discovery and problem solving. In fact, from this point of view, *learning is thinking.* Instead of learning "facts" and then using them in thinking, we can learn "facts" by thinking. This doctrine means that learning should be characterized by insight, and it sharply condemns the traditional practice in arithmetic of having children memorize certain operations in abstract form, in order to apply them in verbal problems afterwards. This practice is probably in considerable part responsible for the difficulty children have in determining what process, or combination of processes, they should use in problem situations.

Instead of authoritatively identifying correct responses for children, courageous teachers are now encouraging active exploration and discovery and self-directed learning. They do so, however, in direct defiance of connectionist dogma. Self-activity may produce errors, and the bond psychologist wants the child to avoid error, for Thorndike has found that (contrary to his earlier statement of the law of effect) the occurrence of a wrong response, even followed by punishment, makes the probability of its occurrence greater, rather than less. So Thorndike admonishes that "the attainment of active rather than passive learning at the cost of practice in error may often be a bad bargain. . . . The almost universal tolerance of imperfect learning in the early treatment of a topic, leaving it to be improved by the gradual elimination of errors in later treatments is probably unsound and certainly risky."[24] This pronouncement places a premium on instruction rather than pupil activity, and stresses authoritative identification by the teacher rather than active discovery by the learner.

Before accepting this advice, however, one should remember that Thorndike's subjects were adults rather than children, and that there was no possibility of learning the tasks in the experiment meaningfully. The correct choices were always arbitrarily determined by the observer. Furthermore, the studies of McConnell,[25] Thiele,[25] and Brownell, all of which stressed pupil discovery and meaningful generalization, gave results which either indicated that active learning produced no undesirable effects or showed that it was decidedly advantageous.

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[24] E. L. Thorndike, *The Psychology of Wants, Interests and Attitudes,* p. 147. D. Appleton-Century Co., New York. 1935.

[25] To be considered below.

Meaning the basis of transfer. Finally, meaningful learning is the key to the transfer of training. A theory of learning as formation of specific bonds with its accompanying doctrine of transfer through identical elements is utterly inadequate to explain how one applies old learning to new conditions, or how he reorganizes experiences creatively. There is a growing body of evidence, on the other hand, which indicates that it is generalization of ideas and processes which facilitates transfer; in fact, it is generalization which makes transfer possible in any important degree. After pointing out the limitations of the doctrine of identical elements, Mursell contends that "in seeking for the true and authentic similarities between two interactive patterns we must look not to their constituent elements but to their *central meanings*." For the greatest transfer to occur, he declares, "a hierarchy of learnings . . . should grow out of one another so that, as the pupil moves along, meanings become more precise, more articulate, more highly differentiated, and at the same time more generalized."[26]

The contribution of meaning and generalization to transfer in arithmetic has been studied by McConnell and Thiele, among others. In McConnell's investigation, one large group of second grade pupils learned the addition and subtraction combinations by procedures which emphasized discovery, organization, and generalization. Another group engaged in activities which stressed authoritative identification, mixed practice, and specific drill. The experiment lasted approximately eight months. During this period, three tests of transfer to untaught processes were administered, and a fourth was included in the final battery. The differences on all four tests favored the meaningful procedures, although only one was statistically significant.[27]

Thiele's results were more conclusive. He compared the learning of 100 addition facts by the methods of specific repetition and meaningful generalization. At the end of the experiment, Thiele administered a transfer test composed of 30 addition examples each of which contained one addend larger than 10. The results showed a mean difference of four examples in favor of the pupils who had used the generalization method. The critical ratio of this difference was 7.4. One must agree with Thiele that "strong evidence is presented by this study to support the faith of those who would make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence."[28] The importance of generalization for learning, retention, and transfer is so great that an entire chapter of this Yearbook is devoted to it.

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[26] J. L. Mursell, *Educational Psychology*, pp. 251-252. W. W. Norton and Company, New York. 1939.

[27] T. R. McConnell, *Discovery Versus Authoritative Identification in the Learning of Children*, pp. 13-62. University of Iowa Studies in Education, Vol. IX, No. 5, September 15, 1934.

There was a time when we looked upon transfer of training as nice to have but so extremely difficult to get that the school should rest its case upon the pupils' acquisition of specific adjustments to specific situations. Now we are recognizing the force of Judd's statement that the "end and goal of all education is the development of systems of ideas which can be carried over from the situations in which they were acquired to other situations." In fact, we now realize that transfer provides the only indubitable evidence that learning actually has taken place.

IMPLICATIONS FOR EXPERIMENTATION

Relation of learning theory to experimental design. The principles of learning discussed above should activate an entirely new experimental program in the psychology of arithmetic. They should also prompt a critical scrutiny of previous investigations. Such an examination would reveal, first of all, that experimental results are functions of the theory of learning which dictated the instructional procedures and the learning activities of pupils. Research on the difficulty of the number facts illustrates this relationship excellently. In the Knight-Behrens study, for example, pupils supposedly learned the number facts by sheer repetition.[29] Furthermore, the facts were presented in mixed order, and only one arrangement was used. Both recent theory and recent research have indicated that the relative difficulty of number relationships is influenced, first, by the order in which they are presented; second, by the way in which they are grouped for instruction; and third, by the method by which pupils learn them. Early difficulty and error studies ranked the zero facts rather high in difficulty. Meaningful methods of learning have demonstrated that they are not especially difficult, and that with the aid of proper generalization, they can be learned as a group without extended drill on the individual items.

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[28] C. L. Thiele, *The Contribution of Generalization to the Learning of the Addition Facts.* Contributions to Education, No. 763. Bureau of Publications, Teachers College, Columbia University, New York. 1938.

[29] F. B. Knight and M. S. Behrens, *The Learning of the 100 Addition Combinations and the 100 Subtraction Combinations.* Longmans, Green and Company, New York. 1928.

A very recent investigation on the comparative difficulty of number facts made repetition the maid-of-all-work in learning.[30] The children learned the combinations by playing the game of Add-0, in which, according to the author, "the correct combinations and answers are always before the children, minimizing the possibilities of establishing incorrect responses. Accuracy is checked and speed is regulated in an effort to control undesirable habits of computation." In the light of his findings, the author advised that "the teacher should not assume that the zero combinations will be learned by inference; they should be taught as any of the other 100 combinations." Of course, the method of instruction did nothing whatever to facilitate inferential and generalized behavior, and probably since Add-0 regulated speed "in an effort to control undesirable habits of computation," the teaching actually frustrated the pupils' own attempts at intelligent apprehension.

How orthodox connectionist principles thoroughly dominated this investigator's thinking and procedure and influenced his results is revealed by his final words of advice to teachers:

> As the size of the addend seems to be the general factor in causing differences in the difficulty ranking, we wonder if the children are not computing the sums by physical or mental counting, a crutch which is probably developed in the child while building the number concepts. Psychologically, the child should be able to learn $5 + 4 = 9$ as easily as $2 + 3 = 5$, but this is not the case according to the investigations of combination difficulties. *It might be more economical first to teach the child to memorize the combinations, and later develop the number concepts.* (Italics are the present writer's.)

[30] L. R. Wheeler, "A Comparative Study of the Difficulty of the 100 Addition Combinations," *Journal of Genetic Psychology*, 54 : 295-312, June, 1939.

Results of investigations of the level of mental maturity necessary for the economical learning of arithmetical processes (such as the widely discussed findings of the Committee of Seven) are probably also functionally related to the method by which the subjects learned these processes. It is conceivable that new studies in which meaningful teaching and learning activities were emphasized would yield substantially different findings.[31]

Need for new types of research. As Brownell has suggested, we need a whole new series of studies to explore the problem of how pupils actually learn. Much of our instructional research, while useful, fails to get at that fundamental problem. Statistical differences between end tests, accompanied by descriptions of intervening *overt* responses of teachers and pupils, do not reveal the critical aspects of the learners' behavior. Brownell's studies have produced ample evidence that what passes for repetition of the same thing disguises an underlying process which is the real cue to the nature and maturity of the pupils' learning.

We also need a new type of error study which is not content with the tabulation of incorrect responses, but which attempts to relate the nature of error to the way in which pupils have learned and to the fashion in which they might learn most economically.

Finally, we must recognize that skill in computation is not the only or even the most important outcome of learning in arithmetic, but that growth in the ability to think quantitatively is a primary objective. Accepting this point of view means devising new instructional activities and also developing new methods of measurement or appraisal.

SUMMARY

Those who are conducting research in arithmetic should realize how vigorously many psychologists are disputing the validity of the principal connectionist maxims. The issues are clear-cut. The newer point of view emphasizes relatedness rather than itemization. It stresses generalization instead of extreme specificity. It conceives of learning as a meaningful, not a mechanical process. It considers understanding more important than mere repetition or drill. It looks upon learning as a developmental process, not one of fixation of stereotyped reactions. It encourages discovery and problem solving rather than rote learning and parrot-like repetition. It is within the matrix of these issues that the new research in arithmetic should be conducted.

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[31] W. A. Brownell, "Readiness and the Arithmetic Curriculum," *Elementary School Journal*, 38 : 344-354, January, 1938.

Chapter XII

QUESTIONS FOR THE TEACHER OF ARITHMETIC

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This chapter contains questions which, the writer hopes, teachers of arithmetic will ask themselves. It is not the purpose of this proposed self-questioning to yield a rating or an appraisal of one's fitness for teaching arithmetic. As a matter of fact, many excellent teachers of arithmetic may score relatively low on these questions, and many poor teachers may score high, since no evidence is available that any of the items of knowledge are functionally related to success in the classroom.

These questions are designed then, not for self-evaluation. Rather, they are intended as "shockers." Teachers, even good teachers, can become complacent about their professional equipment. Some teachers may find in their inability to answer the questions a stimulus and a guide for the reading and study which they may have neglected.

There is another kind of reader who can profit from the self-examination provided in this chapter. This is the person who is in charge of courses on the teaching of arithmetic. Such a person might well check the content of his course against these questions, not with the idea of including in his course all or even most of the items in this list, but with the idea, as in the case of the teachers just mentioned, of stimulating himself to think through again the purposes and the content of his course.

Two important limitations should be recognized with respect to this list of questions. The first is that these questions represent an analysis of the complicated total act that we know as teaching. No one, least of all the writer, likes to think that *any* list of specific items of knowledge or of specific teacher activities can be regarded as equivalent to the complete act of teaching. In a real sense the break-down of teaching into specifics destroys the thing which is analyzed. Yet, on the other hand, values can accrue from lists of specifics. The danger in dealing with them lies in failure to recognize that they *are* specifics. Once one recognizes this fact, one can deal with them without fear of distorted emphasis.

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[1] With the collaboration of J. T. Johnson and B. A. Sueltz.

The second limitation is one of scope. The questions relate only to the teacher's academic and professional knowledge of arithmetic. All those important human elements, such as the teacher-pupil relationship, emotion, attitudes and appreciations, and personality growth, while present in all school work, have not been included in this series of questions. Good teachers not only recognize their import but are alert to situations which may build desirable personality and emotional traits as well as to those situations that have a derogatory effect.

ACADEMIC BACKGROUND

History of arithmetic. Historical perspective enriches the contemplation of current conditions and the possibilities of future development. A study of the evolution of arithmetic as a field of knowledge and as an integral part of the modern school curriculum should prepare the teacher for more intelligent interpretation of the present arithmetical program and for more critical appraisal of any suggestions for future revision of this program.

(1) Am I acquainted with the language and limitations of some of the more significant primitive methods of counting?

(2) Am I acquainted with the relation between the evolution of number systems and the evolution of race culture?

(3) Do I know how the concept of number became abstract through being separated from counted objects?

(4) Do I know the distinguishing characteristics of the Egyptian, Babylonian, Roman, and Greek number systems?

(5) Do I fully appreciate the advantages of our number system over each system listed in (4) above?

(6) Do I understand why our numbers are most properly called Hindu-Arabic *numerals?*

(7) Do I have a clear understanding of why calculation could not show much progress with the early number systems?

(8) Am I familiar with the differences that exist today in the numeration of large numbers? For example, what is the distinction in meaning of *one billion* in England and in the United States?

(9) Am I acquainted with the evolution of the modern methods of the processes of addition, subtraction, multiplication, and division?

(10) Do I appreciate the significance of the printing press in the stabilization of number symbols?

(11) Do I know the origin of fractional numbers and the different types of situations they describe?

(12) Am I acquainted with the evolution of modern fractions and the difficulties intrinsic in different ancient symbolisms?

(13) Am I familiar with the evolution of the modern notation for decimal fractions?

(14) Do I know the historical relation of denominate numbers and fractions?

(15) Am I familiar with the history of our calendar as a measure of time?

(16) Do I know the names, contributions, and approximate periods of at least ten contributors to arithmetic lore?

(17) Do I know something of the origin and development of the English system of weights and measures?

(18) Do I know something of the origin and development of the metric system of weights and measures?

(19) Do I know something of the history and development of calculating machines from the abacus to the slide rule and modern calculating machines?

Rationale of arithmetic. One of the most important of all instructional responsibilities is that of making the learning process meaningful to the pupil. The *why* of any process or technique is most frequently of just as great significance as the *how*. Every teacher of arithmetic should thoroughly understand the rationale of at least that arithmetical content with which he must deal in his teaching.

(1) Have I an appreciation of number as a symbolic language?

(2) Do I have an appreciation of the significance and limitations of counting?

(3) Do I know the distinction between cardinal and ordinal numbers and the function of each in our scheme of numeration?

(4) Have I a clear understanding of the implications of such classifications of number as prime, even, odd, real, complex, etc.?

(5) Do I understand the use of zero as a number symbol, as in $6-6=0$, $3\times 0=0$?

(6) Do I have an appreciation of zero as a symbol to indicate the empty column, as in 506?

(7) Do I understand the full significance of the use of zero as a point on a number scale; for example, as the zero point of a thermometer or as the zero line (base line) of a bar graph?

(8) Do I fully appreciate the importance of *place value?*

(9) Do I know the significant characteristics of a decimal system of numeration?

(10) Do I fully appreciate the advantages and disadvantages of a decimal system of numeration?

(11) Do I appreciate the significance of the statement: "That mankind adopted the decimal system was a physiological accident?" [2]

(12) Do I fully understand the technique of the numeration of large numbers?

(13) Do I understand the principles of scientific notation used in the writing of very large numbers or very small numbers?

(14) Do I know the distinguishing characteristics of other systems of numeration (duodecimal, vigesimal, sexagesimal, etc.)?

(15) Do I appreciate the compactness and simplicity of numerical calculation made possible through the use of the fundamental processes?

(16) Do I have a clear understanding of the fundamental principles which govern carrying, borrowing, and the placement of figures in quotients and partial products?

(17) Do I appreciate the full significance of the etymological meaning of "fraction"?

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[2] Tobias Dantzig, *Number, The Language of Science,* p. 15. The Macmillan Company, New York. 1933.

(18) Do I know the modern uses and limitations of common fractions in both descriptive and computational situations?

(19) Do I have an understanding of decimal fractions as an extension of the decimal system of numeration?

(20) Do I know the comparative superiority and inferiority of common fractions *versus* decimal fractions in both descriptive and computational situations?

(21) Do I know the relations that exist between the fundamental principles and processes of percentage and those of common and decimal fractions?

(22) Do I have an intelligent understanding of the nature and significance of denominate numbers?

(23) Do I know the distinction between arithmetic, numerology, and the theory of numbers?

Social importance of arithmetic. One of the major responsibilities of arithmetical instruction is the use of number concepts and arithmetical techniques in social situations so that the pupil may have the opportunity to appreciate the contribution arithmetic may make to enriched environmental experience.

(1) Do I know any of the evidences that the evolution of number and the accompanying methods of calculation are related to commercial and social progress?

(2) Am I familiar with any of the social demands which produced integers, fractions, negative numbers, etc.?

(3) Am I familiar with any vestiges of number systems, other than the decimal system, in social use today?

(4) Do I understand the social significance of each of the fundamental arithmetical processes?

(5) Do I appreciate the social significance of common and decimal fractions?

(6) Do I appreciate the social significance of percentage?

(7) Do I know the evolution and social backgrounds of computing simple and compound interest?

(8) Do I have the ability to analyze life situations such as banking, farming, sewing, cooking, keeping house, playing games, investments, insurance, etc., for their arithmetical content, and to adapt this content to the grade level at which I teach?

(9) Do I have a satisfactory understanding of the monetary system of the United States?

(10) Am I familiar with the monetary systems of some of the more important foreign countries?

(11) Do I understand the full significance of "standard time" and its applications?

(12) Do I have the ability to sense my own individual needs, direct and indirect, for arithmetic; as in the construction of a personal or business budget, in problems of wise buying, etc.?

(13) Do I have an appreciation of the social importance of the graph as a means for the presentation of numerical information?

(14) Do I have the ability to make applications of arithmetic to other school subjects, such as science, history, geography, health work, etc.?

(15) Do I appreciate the significance of number as an aid to scientific investigation and experimental procedure?

(16) Have I during the past year grown increasingly sensitive to the usefulness of number and to the quantitative element in my own life?

Command of the subject matter of arithmetic. A *sine qua non* of all effective instruction is a thorough knowledge of the subject matter to be taught. For the teacher of arithmetic this implies a familiarity with arithmetical concepts and a proficiency in arithmetical processes which will guarantee a range of arithmetical ability extending above the level at which the teacher is to teach. The teacher should have adequate comprehension of techniques in order that rationalization and generalization of processes will be both natural and intelligible. His informational background should be such as to ensure self-confidence and to supplement well-planned units of instruction with that spontaneity of instruction which is both enlightening and challenging.

(1) Am I able to think in number symbolism?

(2) Am I fully aware of the one-to-one correspondence between objects and positive integers which gives rise to counting?

(3) Am I conscious of the distinction that exists between the process of counting and the actual notation used to record the results of counting?

(4) Am I proficient in the use of the fundamental arithmetical processes?

(5) Have I a command of arithmetical concepts and processes beyond the demands of my immediate grade?

(6) Do I know how to use Roman numerals?

(7) Do I appreciate the efficiency of modern fractional notation?

(8) Do I have a significant understanding of the use of the fundamental processes with common fractions?

(9) Do I have a satisfactory proficiency in working with common fractions?

(10) Am I able to use decimal fractions with ease and intelligence?

(11) Am I able to use decimal fractions as an aid in indicating precision of measurement?

(12) Do I understand the basic vocabulary, principles, and skills of percentage?

(13) Am I able to compute with denominate numbers?

(14) Do I have a satisfactory understanding of the more important tables of weights and measures?

(15) Do I have an appreciation of the relative merits, advantages, and disadvantages of the English and metric systems of weights and measures?

(16) Do I understand the fundamental principles of direct measurement?

(17) Do I understand the fundamental formulas for measurement of distance, area, volume, and capacity from the points of view of derivation and use?

(18) Do I know the fundamentals of scale drawing?

(19) Do I understand the fundamental principles of indirect measurement?

(20) Do I know the fundamentals of ratio and proportion?

(21) Do I know the significant differences between similarity, congruency, and equality of geometric figures?

(22) Do I understand the approximate nature of measurement and the types of errors that are involved?

(23) Do I have an understanding of precision and accuracy of measurement?

(24) Do I understand the use of rounded numbers and how to compute with them?

(25) Do I know how to organize data into a frequency distribution?

(26) Do I have a knowledge of the meaning of the more important statistical terms and how to use them?

(27) Do I know the distinguishing characteristics of the broken line, bar, and circle graphs?

(28) Do I know the fundamental principles that need to be observed and the precautions that should be taken in the construction of these graphs?

(29) Do I know how to interpret each type of graph intelligently?

(30) Am I familiar with the advantages, the disadvantages, and the dangers of misinterpretation of the picture graph?

(31) Do I have sufficient mathematical background to enable me to recognize readily and develop intelligently the mathematical situations that arise extemporaneously in my classroom?

(32) Do I have the informational background necessary to be able to provide opportunities for individual pupils to explore special mathematical interests, and to guide them intelligently in this exploration?

(33) Do I have the ability to recognize relationships that exist in problem situations?

(34) Do I have a clear understanding of the relationships that exist between the fundamental processes?

PROFESSIONAL PREPARATION

Methods of teaching. Teaching arithmetic is a task which will challenge the best efforts of teachers. The efficient teacher must have a thorough knowledge of the subject matter of arithmetic, and skill in the techniques of teaching and in guiding the learning of pupils.

(1) Am I able to recognize and to capitalize upon the many social and economic arithmetical experiences that children have in the home, school, and community?

(2) Am I able to inspire interested students to pursue the study of arithmetic and mathematics to the greatest extent of their ability?

(3) Do I have the inclination and ability to attempt to encourage students to have an interest in the study of arithmetic and mathematics?

(4) Do I have a clear understanding of the reciprocal relationships between arithmetical situations and the fundamental processes of arithmetic?

(5) Do I have a clear-cut idea concerning when, why, and how to rationalize significant arithmetical processes?

(6) Do I have a clear understanding of the functions of practice in teaching arithmetic?

(7) Do I have a sound philosophy concerning the use of "crutches" in learning arithmetic?

(8) Do I know how to encourage original applications of arithmetical processes?

(9) Do I know how to use pupil reports to the best advantage in the teaching of arithmetic?

(10) Do I know how the uses of number and arithmetic may be dramatized?

(11) Do I know the sources of illustrative materials which can be used effectively for motivation purposes?

(12) Do I know how to use a course of study intelligently?

(13) Do I know how to organize large units of instruction?

(14) Do I know how to anticipate and plan the details of learning inherent in large units of instruction?

(15) Do I have sufficient understanding of arithmetic and its implications to be able to lead my pupils to make significant integrations in their thinking?

(16) Do I know how and when to use concrete illustrations as an aid to effective learning?

(17) Do I have an appreciation of the value of objectives to efficient instruction?

(18) Do I know how to make pupils conscious of definite objectives of instruction?

(19) Do I know how to help pupils to distinguish between essential and nonessential data in a problem situation?

(20) Do I know how to teach pupils to estimate and check answers to problems?

(21) Do I have an appreciation of the value of the story element in the presentation of problem material?

Individual differences. Any instruction that is to be effective must reach the individual pupil. Individuals differ in their interests, abilities, and aptitudes. These differences demand an adaptation of instruction designed to discover and provide for individual pupil difficulties. During recent years this adaptation of instruction to individual differences of pupils has been recognized as among the most important of all educational problems.

(1) Do I recognize that pupils differ in rate of learning, in kind and type of learning, in degree of learning, and in interests and aptitudes?

(2) Do I know how to encourage individual initiative on the part of pupils in the study of arithmetic?

(3) Do I know how to develop a feeling of individual responsibility through the use of arithmetic materials?

(4) Do I know how to use the more important recommendations for taking care of individual differences in arithmetic?

(5) Do I know how to locate the specific needs of individual pupils?

(6) Do I know how to lead pupils to develop techniques of self-diagnosis and remedial procedures?

(7) Do I know how to use the best recommended practices for diagnostic and remedial teaching?

(8) Do I know how to eliminate faulty habits and correct inefficient methods of study?

(9) Do I know the recognized areas of special arithmetical difficulty at the grade level on which I teach?

Arithmetic as a school subject. Through the years arithmetic has occupied an important place in the curriculum. In the present era of curriculum change, arithmetic, as an integral part of the school program, has been able to stand successfully against severe critical challenge only through reorganization and new adaptations of its content and revision of many instructional procedures. Such modifications, however, have served to give increased emphasis to the significance of arithmetic as a school subject to the extent that it has lost but little of its previous prestige and has promise of the gain of much future prominence.

(1) Do I have a personal interest in arithmetic in a well-integrated educational program?

(2) Do I have an intelligent understanding and appreciation of the nature and importance of quantitative thinking?

(3) Do I have sufficient informational background to enable me to recognize algebra as a generalization of arithmetic?

(4) Do I have sufficient informational background to enable me to recognize the relationships existing between arithmetic and geometry?

(5) Do I understand how to assist pupils to make the generalizations from arithmetical concepts and techniques to those of algebra and geometry?

(6) Do I have an appreciation of number as an aid to precise and accurate thinking?

(7) Do I have an appreciation of the informational functions of arithmetic?

(8) Do I have an appreciation of the distinction between functional arithmetic and mere computational arithmetic?

Measuring the results of instruction. Efficient evaluation of student progress no longer consists merely in measuring achievement. There must be a continual program for the appraisal of the progress of each individual pupil toward pre-established objectives of instruction. Such a program demands that the efficient teacher be well informed in the techniques, advantages, and disadvantages of various forms of evaluation; the fundamentals of test construction; and the methods of recording and interpreting the results obtained from any technique that may be used to secure a check on pupil progress.

(1) Do I know the fundamental principles of evaluation?

(2) Do I know how to interpret and use intelligently tests of scholastic aptitude and achievement?

(3) Do I have a familiarity with the advantages and disadvantages of the different types of testing?

(4) Am I able to evaluate the strengths and weaknesses of standardized tests in arithmetic?

(5) Do I know the distinguishing characteristics of diagnostic, prognostic, inventory, readiness, and achievement testing?

(6) Do I have an appreciation of the importance of diagnostic, prognostic, inventory, readiness, and achievement testing in an effective instructional program?

(7) Do I know how to use measures of central tendency and of dispersion in analyzing pupil and class progress?

(8) Do I know how to keep accurate records of pupil progress?

(9) Do I know how to use charts and graphs to the best advantage in keeping class and pupil records?

(10) Do I know effective methods of informal evaluation?

Literature on the teaching of arithmetic. In recent years much has been written concerning the improvement of instruction in arithmetic. The teacher of arithmetic should keep conversant with that material which is relevant to his teaching situation. He should know where to find this material and how to evaluate it. He should keep in mind that not all that is published is worthy of serious consideration, but that it is his responsibility to seek that which is of greatest value.

(1) Do I know how and where to find information concerning instructional aids in the teaching of arithmetic?

(2) Am I familiar with recent studies and recommendations with respect to the curriculum content of arithmetic?

(3) Am I familiar with recent studies and recommendations with respect to the grade placement of arithmetic topics?

(4) Am I familiar with the more important current theories of arithmetical instruction?

(5) Do I know where to find digests of experimentation in the teaching of arithmetic?

(6) Am I familiar with the most recent texts, workbooks, tests, and courses of study in arithmetic?

(7) Am I familiar with the most recent books on the teaching of arithmetic?

(8) Do I know the implications of the most significant experiments in the teaching of arithmetic?

(9) Do I know where to find reports on important studies dealing with the teaching of arithmetic?

(10) Do I have access to any of the more important journals that carry discussions of instructional problems in arithmetic?

(11) Am I familiar with the more important movements in elementary education and their significant implications to the better teaching of arithmetic?

Modern psychology of arithmetic. No single formula can be given for the guarantee of efficient instruction in arithmetic. Modern psychology emphasizes the importance of understanding and motivation in learning. Instruction must be meaningful and subject matter must be made interesting. This implies that careful attention must be given to individual interests, abilities, and aptitudes. Furthermore, skills and concepts once developed must be maintained. For widest possible use, the material of instruction must be framed, wherever possible, in the context of the experience of the pupils. There must be careful planning for developmental teaching, maintenance of learning, and expansion of usage.

(1) Do I have a clear understanding of the relationships between concepts, skills, and facts?

(2) Do I know how to judge progress in learning arithmetic?

(3) Do I appreciate the importance of proper grade placement of instructional material?

(4) Do I know how to combine intelligently those learning activities described in the text and those growing out of life experiences?

(5) Do I know how to analyze a class situation to detect the most effective techniques for providing learning experiences?

(6) Do I know how to relate new material to familiar experiences of children?

(7) Do I know how to assist children in relating instructional material to its most pertinent applications?

(8) Do I have a justifiable point of view on the relative value of speed *versus* accuracy in arithmetical computation?

(9) Do I have the ability to distinguish between the weak points and strong points of recommendations for the improvement of arithmetical instruction?

(10) Do I know the importance of *significance, meaning,* and *insight* to intelligent instruction in arithmetic?

(11) Do I know the meaning and implications of arithmetic readiness?

(12) Do I know the most important implications of the theory of transfer of training to the arithmetic curriculum?

(13) Do I know the more important implications of the modern theory of transfer of training to instruction in arithmetic?

(14) Do I know the significance of practice as a factor in learning?

(15) Do I appreciate the importance of interest as a factor in learning?

(16) Do I understand the function of review as an aid to effective learning?

(17) Do I know the distinguishing features and the important values of oral and written exercises as an aid to effective learning?

It is believed that answers to the above questions will aid the conscientious teacher of arithmetic in acquiring a broad perspective of arithmetic in its relation to the educational program of the individual and the instructional techniques of the classroom. Such a perspective should give that teacher a more significant appreciation of the importance of arithmetic as a school subject and aid him as he earnestly strives to meet the challenges of the varied responsibilities of his profession.

Chapter XIII

THE INTERPRETATION OF RESEARCH

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NOT all research relating to arithmetic is trustworthy. Sometimes factors which influence learning have not been identified and as a consequence have not been controlled. Sometimes the experimental problem has been viewed too narrowly; vital relationships have not been recognized; and the significance of the findings is correspondingly restricted. Sometimes important phases of growth and learning have not been included in the evaluation, and the superiority shown for this or that teaching procedure, for example, is much less real than it appears to be. Sometimes technical errors in the prosecution of the study (too few subjects, too short a period of instruction, unsuited learning materials, and the like), and sometimes inadequate or invalid forms of statistical analysis render questionable the conclusions which are drawn. And sometimes bias on the part of the investigator unintentionally determines crucial aspects of the research technique and colors the interpretation of the results which are obtained.

These limitations of research and other limitations which might be mentioned are not new, nor are they confined to research in the field of arithmetic. They are referred to here because they need to be kept continually in mind as one reads the large and growing research literature. With regard to all research two extreme and equally objectionable attitudes are sometimes encountered. On the one hand is the attitude of those individuals who tend to magnify the limitations which attach to much arithmetic research and accordingly to reject the whole of it as inconclusive or even misleading. For such persons this chapter can obviously serve no useful end; what they need is not warnings but constructive evidence of the values of research, and there is no space for the presentation of this evidence.

This chapter is, rather, intended to combat the second attitude alluded to just above, namely, the attitude of extreme readiness to accept anything which is published as a report of research. The number of such uncritical readers of research is by no means small, and the number of errors which, in their eagerness to "apply" research findings, they can introduce into schoolroom practice is certainly not negligible. The writers of this chapter would not lessen anyone's conviction that research must provide us the final answers to the many questions relating to the teaching of arithmetic. (They would not do so, for they themselves share this conviction.) On the other hand, such faith in the ultimate contribution of reasearch must not blind us to the imperfections of research which is not competently done and not wisely interpreted.

Were it practicable and desirable to do so, this chapter would contain a brief list of criteria for evaluating research. Attempts have been made to formulate such criteria, and various lists are available in print. To be manageable, such a list must be short, consisting in perhaps five or six criteria but certainly no more than ten. A list that is too long and detailed defeats its own purpose. The application of each criterion in a long list constitutes no mean research investigation in itself, and may well carry the critic far afield from the study at hand. The situation in this case does not differ materially from that of the high school or college student who tries to study by "applying" some list of thirty or forty principles on "how to study," and becomes so busy in the "applying" that he never gets around to studying. On the other hand, if the number of criteria is held to a few, the criteria become either trite and obvious or involved and obscure. In either case the problem of identifying each particular criterion with all the relevant phases of the research is no easy one.

Rather than undertaking to list criteria, whether few or many, the writers have chosen to illustrate in two ways the need for care in reading research. The section which immediately follows shows how the peculiar point of view one entertains with respect to arithmetic affects both the prosecution and the interpretation of research. The last section of the chapter is, as it were, a "case study" of the research on division. It is designed to show how research findings and recommendations are conditioned by the technical aspects of experimentation.

POINT OF VIEW AND ITS RELATION TO RESEARCH

It is possible to hold one of several views with regard to arithmetic. One may argue that the purpose of arithmetic is to develop efficiency in computation or to develop expertness in quantitative thinking. One may view arithmetic as a drill subject or as a logical system which must be understood. One may regard arithmetic as a form of pure mathematics or as a practical means of improving our adjustment to a quantitative culture. Or, one may reject these dichotomies and, refusing to think in terms of "either" and "or," one may arrive at some eclectic position, such as that sponsored by this Yearbook Committee.

Effect produced by conception of arithmetic. Now the conception which one entertains with respect to the function of elementary arithmetic influences research at many points. In the first place, it predisposes one to attack some problems and to disregard others. In the second place, it largely predetermines the technical procedure which one organizes and more especially the kinds of measures employed. In the third place, it may prejudice the investigator in the interpretation he places upon his data and the reader in the significance which he attaches to the research report he is analyzing. These three statements will be clarified by the examples offered below.

It is safe to say that the dominant conception up to some fifteen or twenty years ago was that of arithmetic as a drill subject, and this conception is reflected in the problems then undertaken for research study. Fully three-fourths of research studies prior to 1920 or 1925 dealt with computation. Such studies as dealt with "problem solving" virtually reduced this process to an extension or an "application" of computational skill. Little if any research was concerned with the learning process as such or with securing evidence that children did or did not see sense and value in what they acquired. And the research measures collected in these studies were restricted to those of efficiency, which is to say, measures of rate and accuracy of work.

Consider, for example, the research on drill as a method of teaching (or on repetition as a method of learning). Three-minute periods were compared with longer and shorter periods, and the crucial data were based not on measures of understanding by the child, but on speed and correctness of answer. Or, consider the research on methods of teaching subtraction, whether by equal additions or by decomposition. Here again the data were based upon the comparative merits of the two methods, in increasing the number of correct answers and the quickness with which they were obtained.[1] Such questions as the following were seldom asked, and if they were asked no answers could be determined from the collected data: Which method appears to be more sensible to children, that is, which method is more readily understood? Which method develops more completely the meaning of subtraction as a process? Which method has the greater possibilities for fruitful transfer to later learning, as in complicated subtraction and problem solving of a functional sort?[2]

The criticism here is not that measures of rate and accuracy are valueless, nor yet that drill is unnecessary. The point to be observed is that a particular conception of arithmetic dictated the selection of the research problem and the kinds of measure obtained. And, in the opinion of this Committee, this conception of arithmetic is inadequate: It omits any consideration either of greater comprehension of what is learned or of enhanced appreciation of its worth. Accordingly, this early work on drill and on methods of subtraction is variously to be assessed. It is important according to the weight attached to computational efficiency as an arithmetical outcome.

The preceding pages have presented one aspect of the relationship which obtains between point of view with respect to arithmetic on the one hand and arithmetic research on the other, namely, that one's conception of arithmetic as a school subject tends to predetermine the problems one will investigate and the measures one will obtain. A second aspect of this relationship is found in the area of learning.

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[1] Of course other measures were sometimes secured, such as the persistence of the method taught, and the preferences of teachers for one or the other of the methods; but the fact remains that the crucial measures were usually regarded as those of rate and accuracy of pupils' work.

[2] These questions, asked in the midst of a controversy by Knight, *et al.*, have never been made the subject of directly related research. See: F. B. Knight, G. M. Ruch, and O. S. Lutes, "How Shall Subtraction Be Taught?" *Journal of Educational Research*, 11 : 157-168, March, 1925.

Effect produced by conception of learning. The arithmetic research of twenty and even fifteen years ago reveals very little attention to learning, apart, as has been mentioned, from measures of rate and accuracy of work. The principal task of research was to "identify the bonds which are to be formed," and the task of teaching, to see that these bonds were established. Following this lead, research gave us elaborate analyses of facts to be acquired, unit skills and steps to be mastered, types of problems to be solved, and specific cues to be learned. Research did not tell us *how* children gain control over the multitudinous elements resulting from these analyses, for research was not concerned with these matters. Learning was thought of as an exceedingly simple process of making connections, or bonds, or associations. If one learned, one showed thereby that one had formed the needed connections, and vice versa. The tendency was to think of learning as an all-or-none affair; it was as if children learned at once what they were supposed to learn, apart from drill, or they did not learn at all.

The research engaged in under this conception of learning did not prove very helpful to teachers. Even the many error studies fell short: they pointed out the places where children encountered difficulty and something of the form of the difficulty, but they did not show teachers (or pupils) *why* the errors were made. Nor indeed could they do so, so long as learning was conceived in such simple, uncomplicated terms, so long as it was measured as it was measured, and so long as important individual variations in learning were hidden in the statistical averages of group trends.

However, the reports of this comparatively old research survive in the literature and they must be interpreted in the light of *some* conception of learning. If one accepts the conception of learning which prevailed a quarter of a century ago, his evaluation of the early learning" studies will be of one sort; if, however, he accepts the conception which now seems to be more adequate and fruitful, his evaluation will be something quite different.[3]

In this connection certain guides to thinking may be suggestive. As one reads reports of learning studies in arithmetic, one may well keep in mind such questions as the following, questions which, incidentally, relate not to the technicalities but to the "common sense" of research: (1) What is the writer's point of view with respect to arithmetic? Does it agree with modern conceptions of the purposes of the subject? (2) What program of instruction and of learning activities was adopted? Was it conducive to or subversive of sound, intelligent, and economical learning? (3) Were the measures which were used for evaluation relevant and adequate? (4) Do the conclusions follow from the data, and are they in line with the best in thought and experience? The answers to these questions will generally be found to be most illuminating.

Point of view and interpretation of research. There is space for but two more examples of the way in which point of view influences the interpretation of research. A few years ago a new monograph reporting a research study in arithmetic was reviewed by two different persons. The one person commented favorably but not enthusiastically upon the study itself, which showed quantitatively the superiority of a certain instructional device over instruction without the device. When, however, he came to the last chapter, in which the investigator discussed the implications of his study for arithmetic teaching and learning in general, this reviewer became more enthusiastic. The other reviewer bestowed his approval in quite a different manner. To him the contribution was the evidence of the value of the particular device, and the last chapter "should never have been written." The same dissimilarity of evaluation characterized the reception accorded an earlier study, in which the drill provisions in several arithmetic textbook series were compared. The sponsor of this study referred to it as a "brilliant piece of scholarly research"; another student of arithmetic described the study as a "laborious effort to no end, a monument to futility."

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[3] The very cautions and warnings which are the subjects of this chapter need themselves to be taken with some reservations. For example, the criticisms of the research of a quarter-century ago, particularly of the drill and error studies, should not be interpreted as implying that this research was valueless and served no end. Such is not the case, even though these investigations contribute little to our thinking about arithmetic in the 1940's. Certainly they made important contributions historically both to the evolution of our view of arithmetic today and to the procedures which we employ with new confidence. The following quotation from a letter from Professor Brueckner brings out this point very well indeed: "The cumulative and developmental nature of research should be recognized. The error studies revealed a condition that shocked many of us. They led to studies of method, reliability, diagnosis, methods of learning, etc. The fact that these early studies did lead to this important series of developments which are even now being extended needs to be emphasized. Studies now being made of learning are developing new technics in this field which yield problems that will instigate new studies. Research is a continuing process."

In both instances the differing evaluations sprang from unlike conceptions of the function of arithmetic in the elementary school, and the two evaluations are equally valid, once one grants validity to each basic conception. The appraised worth of the two studies, as of research in general, is not exclusively determined by purely intrinsic features of merit or of weakness, but by the point of view with which the reader approaches his evaluation.

Enough has been said, it is hoped, to justify the statement that conceptions of arithmetic and of the learning process have important bearings upon the research that is done and upon the interpretation that is given research findings. The implications of this statement are not confined to the producer of research, though it is chiefly these implications which may seem to have been stressed; its implications are no less vital and far-reaching for the consumer of research. As he reads research reports, the latter needs always to note the conceptions which have controlled the investigator; and he needs also to have a clear formulation of his own conceptions. He must know what he himself thinks about the function of arithmetic in the elementary school and about the way children learn most soundly, and he must be on the alert to see wherein the research in question confirms and implements his conception and wherein it requires modifications in his conception. A clear-cut view of the purposes of arithmetic and yet a view which is amenable to change as change is called for—such a view enables one to detect both the strengths and weaknesses of research and to distinguish between what is sound and what is erroneous and misleading.

THE RESEARCH ON DIVISION

It was stated that in this chapter two illustrations of the need for care in interpreting research were to be given. The first, already discussed, has to do with the effect of point of view. The second illustration relates to the research on a particular topic, namely, division. This research is selected as exemplifying particularly well the contingent character of research findings, their dependence upon the special technical aspects of investigation, and their tentativeness because of untested hypotheses and uncontrolled factors. Not all the research within even this narrow field can be canvassed. That, for example, on the errors made by children, on the relative merits of different methods of teaching, and on the comparative values of different procedures for estimating quotient figures must be omitted because of space limitations. Attention here is confined to two questions: first, the form to be used at the outset in teaching division and, second, the grade placement of "long" division.

Research on the form used in division. "Form" refers to the "long" (that is, division completely worked out) or the "short" algorism in division when the divisor is a one-figure number. John[4] showed that the long form is superior to the short form for accuracy but not for speed. The published report does not state whether the numerical difference found between the two experimental groups in each of these traits is statistically reliable. Regardless of this omission, the results are open to question because only fourteen pairs of subjects took part in the experiment. John further showed that after division with a two-figure divisor has been taught, then division with a one-figure divisor is learned in the long form with still greater accuracy and speed than in the short form. If the groups used in the study had been large enough, this last finding would have been extremely significant because, unlike other studies, this one involved a comparatively long range of time in the learning of division. The relation between division with one- and two-figure

[4] Lenore John, "The Effect of Using the Long-Division Form in Teaching Division by One-Digit Numbers," *Elementary School Journal,* 30 : 675-692, May, 1930

divisors deserves serious consideration, something which it has not yet received elsewhere.

Olander and Sharp[5] reported an investigation which included an adequate sampling of subjects (1,265 pupils) to warrant reliable measures. They concluded that, on the whole, the long form is superior to the short form of division, but that the latter is preferable (*a*) for easy examples and (*b*) for bright pupils. These last conclusions may be challenged on the grounds, first, that the length of the experiment (apparently it was of short duration) is not reported and, second, that no criteria for "easy" as contrasted with "difficult" examples are offered. These investigators did, however, advance our understanding by calling attention to possibly important factors (quality of pupil, ease or difficulty of example) which theretofore had been neglected.

Grossnickle[6] showed that the results obtained by the long form of division were significantly superior to those obtained by the short form for difficult examples. A total of 2,365 pupils in eight different grades were used in this study. For easy examples (defined as those in which there is no borrowing in subtraction and the remainder is not in excess of 3), less time is needed to find the answer by the short form, but greater accuracy always results with the long form. (The difference in time was not statistically reliable; that in accuracy was.) Since the elements of time available for work, of accuracy, and of degree of difficulty of the example were all considered, and since an adequate number of subjects were used, the findings of this investigation should probably be accepted as valid and reliable.

John and Olander and Sharp (in the references cited) recommended teaching the short form of division later and as a short-cut procedure, but Grossnickle recommended that only the long form should be taught. There is no experimental evidence to justify either proposal. In each case the investigator injected his own opinion as to the value of the shorter procedure. The true worth of a short cut in division with a one-figure divisor represents an unexplored question in the teaching of arithmetic.

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[5] Herbert T. Olander and E. Preston Sharp, "Long Division Versus Short Division," *Journal of Educational Research*, 26 : 6-11, September, 1932.

[6] Foster E. Grossnickle, "An Experiment with a One-Figure Divisor in Short and Long Division," *Elementary School Journal*, 34 : 496-506, 590-599, March, April, 1934.

The relative merits of the two forms of division will be known only when investigation reveals a real difference between groups of pupils thoroughly taught division by the two forms. None of the studies thus far reported, with the exception of that by John, is a learning study, that is to say, in all but this one study the experimental subjects had been taught to divide by a one-figure number before the investigation as such was undertaken. In the case of the one learning research (John) the period of study for both the long-form and the short-form groups lasted but a week. Unless both groups had earlier received some instruction in division, it is hardly possible that they could have carried their learning to anything like the required degree of mastery. There is great need for a comprehensive and long-range study, in which a serious attempt will be made to determine, not only which form yields greater efficiency, but also which form results in greater insight. In the end, it is not at all unlikely one form may be found superior for some pupils, and the other form superior for other pupils.

Research on grade placement. The second area of division research to be considered here deals with the grade placement of the topic. Not much scientific work has been done in this connection. One of the first and most widely known investigations was conducted by the Committee of Seven under the chairmanship of Washburne.[7] This Committee determined placement of topics in terms (*a*) of foundations tests, which supposedly contained samples of all prerequisite skills and concepts, and (*b*) in terms of Mental Age. A Mental Age of twelve years seven months or more was set as the minimum maturity level for the learning of division with a two-figure divisor, at least of the more difficult kinds. In other words, long division according to the Committee of Seven belongs in the latter part of grade five or the first part of grade six.

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[7] Carleton Washburne, "Mental Age and the Arithmetic Curriculum: A Summary of the Committee of Seven Grade Placement Investigations to Date," *Journal of Educational Research,* 23 : 210-231, March, 1931. Also, by the same author: "The Values, Limitations and Applications of the Findings of the Committee of Seven," *Journal of Educational Research,* 29 : 694-707, May, 1936. "The Work of the Committee of Seven in Grade Placement in Arithmetic," *Thirty-eighth Yearbook of the National Society for the Study of Education, Part I,* Chapter XVI, 1939. This last reference contains the last of this Committee's several summaries.

The findings of the Committee of Seven a[illegible]pen to question and must be modified in the light of exp[illegible]mental evidence. Brownell[8] showed points of weakness in the [illegible]earch of this Committee, and both Brownell[9] and Dickey[10] [illegible]ve emphasized the point that readiness for the topics of arith[illegible]etic is not solely or perhaps even largely dependent upon Mental Age as measured by intelligence tests. This view has been confirmed by Grossnickle's study.[11] In the first named investigation, pupils were subjected to a program of instruction which was characterized by unusually careful diagnosis and corrective practice, and under these conditions pupils with a Mental Age about two years less than the standard of the Committee of Seven learned to divide by a two-digit divisor and attained a higher degree of accuracy than that employed by the Committee.

Division is a complex process because it is a synthesis of many factors. Hence, the placement of division as a whole in this or that grade or at this or that age or at this or that maturity level is impossible. The example, $9\overline{)2112}$, because of the difficulty in estimation and in subtraction, is more difficult than the example, $21\overline{)672}$, in spite of the fact that in the former the divisor is a single digit instead of a two-place number. The example, $16\overline{)912}$, is very much more difficult than the example, $203\overline{)8729}$, because corrections of estimation are needed in the first example but not in the second example. These examples illustrate why it is impossible to assign division to any point in the grades or to any maturity level merely in terms of the number of figures in the divisor. If it is stated that a Mental Age of about thirteen years is needed to learn to divide with a two-figure divisor, this statement must be qualified by specifications as to the kind of example referred to.

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[8] William A. Brownell, "A Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics," *Elementary School Journal*, 38 : 495-508, March, 1938.

[9] William A. Brownell, "Readiness and the Arithmetic Curriculum," *Elementary School Journal*. 38 : 344-354, January, 1938.

[10] John W. Dickey, "Readiness for Arithmetic," *Elementary School Journal*, 40 : 592-598, April, 1940.

[11] Foster E. Grossnickle, "An Experiment with Two Methods of Estimation of the Quotient," *Elementary School Journal*, 37 : 668-677, May, 1937; also "How the Method of Scoring a Test in Division Affects the Score," *Elementary School Journal*, 40 : 366-370, January, 1940.

Consider in this connection the range in difficulty between the following examples with two-digit divisors: $21/\overline{672}$ and $16/\overline{912}$.

Brueckner and Melbye[12] have shown that the degree of accomplishment in division is dependent upon the difficulty of finding the true quotient figure, a point made more or less incidentally in the preceding paragraph. If the estimated quotient figure is the true one, the accomplishment of a class in division is much greater than when the estimated figure has to be corrected, a finding which led them to conclude that the grade placement of division should be partially governed by the difficulty of the example in this respect. And difficulty of examples is in turn dependent not only, as they have shown, upon the number of corrections needed to find the true quotient figure, but also upon the number of figures in the divisor, the number in the quotient, and the size of the guide figure of divisors involving more than one digit.

The finality of these results may be questioned when compared with those of another study. Brueckner and Melbye report that "the data for the zero-in-quotient type show that the index of 25 per cent error, or 75 per cent accuracy, on the whole is not reached before the mental age of 168 months, the same point as for the one-figure quotient examples in which the apparent quotient must be corrected to find the true quotient." On the other hand, Grossnickle[13] analyzed errors made with one-figure divisors and with two-figure divisors.[14] In the first case, the pupils had already learned the process, and the test given was intended to measure their accomplishments, but in the latter case the pupils were just learning the process. When the divisor was a one-figure number, about 13 per cent of all errors in each grade from grades 5-15, inclusive, was due to the use of zero. However, when the divisor was a two-figure number, the per cent of errors resulting from the use of zero was 0.3. The zero-type of example was among the easiest which this group encountered in division, and the reason probably lay in the way in which the function of zero was taught and in the way in which the examples containing zero were graded for instructional purposes. The function concept of zero as a place holder in the quotient had been stressed with these pupils; as a consequence they did not merely juggle zeros, but rather understood why zero must be used to give place value to quotient figures. Therefore, to the factors already mentioned as affecting grade placement must be added the method of teaching employed and the degree of understanding of the process attained.

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[12] Leo J. Brueckner and Harvey O. Melbye, "Relative Difficulty of Examples in Division with Two-Figure Divisors," *Journal of Educational Research*, 33 : 401-414, February, 1940.

[13] Foster E. Grossnickle, "Errors and Questionable Habits of Work in Long Division with a One-Figure Divisor," *Journal of Educational Research*, 29 : 355-368, January, 1936.

[14] Foster E. Grossnickle, "Constancy of Error in Learning Division with a Two-Figure Divisor," *Journal of Educational Research*, 33 : 189-196, November, 1939.

All the factors have not yet been named which must be taken into account experimentally in deciding the grade in which division is to be taught. Research, chiefly that of Grossnickle, has shown that children *can* learn division in the fourth grade, but this does not mean that the topic *should* be taught in that grade. Another factor which still further complicates the placement of the topic is its social significance for the pupil. If the social significance of division is slight for pupils in the fourth grade but greater for pupils in the sixth grade, other things being equal, the topic may well be reserved for the later grade. The social significance of division as a phase of grade placement is as yet undetermined through research.

A valid answer to the question of the grade placement of division must therefore be based upon consideration of many factors, such as the inherent structure of examples in the process, the method employed for teaching it meaningfully, and the social significance of the topic. There is no evidence to indicate that many of these factors were experimentally considered in the Committee of Seven's placement of division at Mental Age twelve years seven months. Nor have all these factors been duly weighed by other investigators, with the result that as yet we do not confidently know on the basis of research just when the topic (or some part of it) should first be introduced.

IN CONCLUSION

This chapter has been devoted to cautions to be observed in the reading of research. It has been pointed out that investigators are influenced in the selection of their problems, in the organization of their experimental procedures, and in the interpretation of their data, by the views they entertain with respect to the nature of learning and with respect to the purposes of arithmetic in the elementary curriculum. And it has been stated that these conceptions are no less influential in the case of the consumer of research; they determine largely the evaluation he assigns to the research he analyzes. Furthermore, in this chapter part of the research on division has been analyzed to reveal the many facets of the problems which relate to the form in which division is first to be introduced and to the determination of the grade in which the topic is first to be taught.

What has been said in these connections is largely negative, and intentionally so. The writers of the chapter have felt it necessary to stress the dangers inherent in the superficial interpretation of research rather than to stress the values of research. It was assumed that these values would be conceded. But such concessions may go too far, and the potential values of research may be allowed to conceal the weaknesses present in much arithmetic research. The cause of research is not advanced, but rather hindered, by uncritical acceptance of questionable data and doubtful conclusions. Low standards of evaluation can only encourage the production of more mediocre studies. When readers of research recognize the complexity of the problems involved in the teaching of arithmetic and when they refuse to countenance unsound findings, the quality of research must inevitably be improved. Negative comments made now may in the end lead to higher standards of research and to markedly improved research investigations.

Chapter XIV

ONE HUNDRED SELECTED RESEARCH STUDIES

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To make this yearbook more serviceable to the teachers, supervisors, and others for whom it has been prepared, a selected list of research studies in arithmetic has been compiled and is presented herewith. In selecting research studies, various criteria may be employed, among which are (1) validity of conclusions, (2) excellence of technique employed, and (3) effects of the findings on educational practice. There are few research studies which rank high in all three of these respects.

It should be understood that neither the writer nor the other members of the Committee can guarantee the soundness of the conclusions which have been reached in these researches. Some are doubtless more valid than others but, on the whole, the conclusions presented in the studies listed are probably more valid than are the conclusions reached in the average research study in arithmetic. However, the reader will have to form his own conclusion as to the validity of each.

A major value of some researches lies in the fact that they illustrate the use of desirable techniques. The researches in the following list employ various different techniques which may well be used by students who wish to make studies of their own.

The critical reader may contend that some of the studies listed were made so long ago that their present usefulness is doubtful. The writer has deliberately included in the list a few studies which have an important historical value even though they no longer influence educational practice as much as they did in the years immediately following their publication. They are landmarks of educational progress, as it were, and are known to thorough students of the teaching of arithmetic.

1. Bond, Elias A. *A Professional Treatment of the Subject Matter of Arithmetic.* New York: Bureau of Publications, Teachers College, Columbia University, 1934.
2. Bowman, Herbert Lloyd. *The Relation of Reported Preference to Performance in Problem Solving.* University of Missouri Bulletin, Vol. 30, No. 36, Education Series, No. 29. Columbia, Mo.: University of Missouri, 1929.
3. Breed, Frederick S. and Ralston, Alice L. "The Direct and Indirect Methods of Teaching the Addition Combinations." *Elementary School Journal,* 37 : 283-295, December, 1936.
4. Brownell, William A. "An Evaluation of an Arithmetic Crutch." *Journal of Experimental Education,* 2 : 5-34, September, 1933.
5. Brownell, William A. *The Development of Children's Number Ideas in the Primary Grades.* Supplementary Educational Monographs, No. 35. Chicago: Department of Education, University of Chicago, 1928.
6. Brownell, William A. and Chazal, Charlotte B. "The Effects of Premature Drill in Third-Grade Arithmetic." *Journal of Educational Research,* 29 : 17-28, September, 1935.
7. Brownell, William A. and Stretch, L. B. *The Effect of Unfamiliar Settings on Problem-Solving.* Durham, N. C.: Duke University Press, 1931.
8. Brownell, William A. and Watson, Brantley. "The Comparative Worth of Two Diagnostic Techniques in Arithmetic." *Journal of Educational Research,* 29 : 664-676, May, 1936.
9. Brownell, William A. with the assistance of Kuehner, Kenneth G. and Rein, William C. *Learning as Reorganization.* Duke University Research Studies in Education, No. 3. Durham, N. C.: Duke University Press, 1939.
10. Brueckner, Leo J. "An Analysis of the Instructional Practices in Typical Classes in Schools of the United States." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 32-50. New York: Bureau of Publications, Teachers College, Columbia University, 1935.
11. Brueckner, Leo J. "Analysis of Difficulties in Decimals." *Elementary School Journal,* 29 : 32-41, September, 1928.
12. Brueckner, Leo J. "Analysis of Errors in Fractions." *Elementary School Journal,* 28 : 760-770, June, 1928.
13. Brueckner, Leo J. and Elwell, Mary. "Reliability of Diagnosis of Error in Multiplication of Fractions." *Journal of Educational Research,* 26 : 175-185, November, 1932.

14. Brueckner, Leo J. and Melbye, Harvey O. "Relative Difficulty of Types of Examples in Division with Two-Figure Divisors." *Journal of Educational Research,* 33 : 401-414, February, 1940.

15. Buckingham, B. R. "The Additive versus the Take-away Method of Teaching the Subtraction Facts." *Educational Research Bulletin* (Ohio State University), 6 : 265-269, September, 1927.

16. Buckingham, B. R. "Upward versus Downward Addition." *Journal of Educational Research,* 16 : 315-322, December, 1927.

17. Buckingham, B R. and MacLatchy, Josephine. "The Number Abilities of Children When They Enter Grade One." *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Education,* Part II, pp. 473-524. Bloomington, Illinois: Public School Publishing Company, 1930.

18. Burge, Lofton V. "Types of Errors and Questionable Habits of Work in Multiplication," *Elementary School Journal,* 33 : 185-194, November, 1932.

19. Buswell, G. T. and John, Lenore. *The Vocabulary of Arithmetic.* Chicago: The University of Chicago, 1931.

20. Buswell, G. T. with the cooperation of John, Lenore. *Diagnostic Studies in Arithmetic.* Supplementary Educational Monographs, No. 30. Chicago: Department of Education, University of Chicago, 1926.

21. Clapp, Frank L. *The Number Combinations: Their Relative Difficulty and the Frequency of Their Appearance in Textbooks.* Bureau of Educational Research, Bulletin, No. 2. Madison, Wisconsin: University of Wisconsin, 1924.

22. Clark, J. R. and Vincent, E. L. "A Study of the Effect of Checking upon Accuracy in Addition." *Mathematics Teacher,* 19 : 65-71, February, 1926.

23. Connor, William L. and Hawkins, Gertrude C. "What Materials Are Most Useful to Children in Learning to Solve Problems?" *Educational Method,* 16 : 21-29, October, 1936.

24. Dickey, John W. "The Value of Estimating Answers to Arithmetic Problems and Examples." *Elementary School Journal,* 35 : 24-31, September, 1934.

25. Edwards, Arthur. "A Study of Errors in Percentage." *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Education,* Part II, pp. 621-640. Bloomington, Illinois: Public School Publishing Company, 1930.

26. Engelhart, Max D. "The Relative Contribution of Certain Factors to Individual Differences in Problem Solving Ability." *Journal of Experimental Education,* 1 : 19-27, September, 1932.

27. Foran, T. G. "The Reading of Problems in Arithmetic" and "Methods of Teaching Problem Solving in Arithmetic." *Catholic Educational Review,* 31 and 32 : 601-612; 154-164; 269-282, December, 1933, March and May, 1934.

28. Fowlkes, John Guy. "A Report of A Controlled Study of the Learning of Multiplication by Third-Grade Children." *Journal of Educational Research,* 15 : 181-189, March, 1927.

29. Grant, Albert. "An Analysis of the Number Knowledge of First-Grade Pupils According to Levels of Intelligence." *Journal of Experimental Education,* 7 : 63-66, September, 1938.

30. Grossnickle, Foster E. "An Experiment with Two Methods of Estimation of the Quotient." *Elementary School Journal,* 37 : 668-677, May, 1937.

31. Grossnickle, Foster E. "Concepts in Social Arithmetic for the Eighth Grade Level." *Journal of Educational Research,* 30 : 475-488, March, 1937.

32. Grossnickle, Foster E. "The Effectiveness of Checking Subtracting by Addition." *Elementary School Journal,* 38 : 436-441, February, 1938.

33. Grossnickle, Foster E. "Methods of Estimating the Quotient in Long Division Used by Teacher-Training Students." *Elementary School Journal,* 35 : 448-453, February, 1935.

34. Grossnickle, Foster E. "Transfer of Knowledge of Multiplication Facts to Their Use in Long Division." *Journal of Educational Research,* 29 : 677-686, May, 1936.

35. Grossnickle, Foster E. and Snyder, John H. "Constancy of Errors to Basic Facts in the Fundamental Operations in Arithmetic." *Journal of Educational Research,* 32 : 336-344, January, 1939.

36. Gunderson, Agnes G. "Nature and Amount of Arithmetic in Readers for Grades I and II." *Elementary School Journal,* 36 : 527-540, March, 1936.

37. Gunderson, Agnes G. "Number Concepts Held by Seven Year Olds." *The Mathematics Teacher,* 33 : 18-24, January, 1940.

38. Hamilton, Dwight. "An Experiment with an Individual Technique in Sixth Grade Arithmetic." *Journal of Educational Research,* 18 : 339-344, November, 1928.

39. Hanna, Paul R. *Arithmetic Problem Solving: A Study of the*

Relative Effectiveness of Three Methods of Problem Solving. New York: Bureau of Publications, Teachers College, Columbia University, 1929.

40. Hanna, Paul R. *et al.* "Opportunities for the Use of Arithmetic in an Activity Program." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 85-120. New York: Bureau of Publications, Teachers College, Columbia University, 1935.
41. Harap, Henry and Barrett, Ursula. "Experimenting with Real Situations in Third Grade Arithmetic." *Educational Method,* 16 : 188-192, January, 1937.
42. Harap, Henry and Mapes, Charlotte E. "The Learning of Decimals in an Arithmetic Activity Program." *Journal of Educational Research,* 29 : 686-693, May, 1936.
43. Holloway, Harry Vance. *An Experimental Study to Determine the Relative Difficulty of the Elementary Number Combinations in Addition and Multiplication.* Philadelphia: University of Pennsylvania, 1914.
44. Howell, Henry Budd. *A Foundation Study in the Pedagogy of Arithmetic.* New York: The Macmillan Company, 1914.
45. Hydle, L. L. and Clapp, Frank L. *Elements of Difficulty in the Interpretation of Concrete Problems in Arithmetic.* Bureau of Educational Research Bulletin, No. 9. Madison, Wisconsin: University of Wisconsin, 1927.
46. John, Lenore. "Difficulties in Solving Problems in Arithmetic." *Elementary School Journal,* 31 : 202-215, November, 1930.
47. John, Lenore. "The Effect of Using the Long Division Form in Teaching Division by One-Digit Numbers." *Elementary School Journal,* 30 : 672-692, May, 1930.
48. Johnson, J. T. "A Survey of the Use of Metric Measures." *The Role of Research in Educational Progress,* pp. 230-234. Official Report of the American Educational Research Association, 1937. Washington: American Educational Research Association of National Education Association, 1937.
49. Johnson, J. T. "Economy in Teaching Arithmetic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 121-156. New York: Bureau of Publications, Teachers College, Columbia University, 1935.
50. Johnson, J. T. *The Relative Merits of the Three Methods of Subtraction.* New York: Bureau of Publications, Teachers College, Columbia University, 1938.

51. Judd, Charles Hubbard. *Psychological Analysis of the Fundamentals of Arithmetic.* Supplementary Educational Monographs, No. 32. Chicago: Department of Education, University of Chicago, 1927.

52. Judd, Romie Dustin and Morton, R. L. "Current Practices in Teacher Training Courses in Arithmetic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 157-172. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

53. Knight, F. B. and Behrens, Minnie S. *The Learning of the 100 Addition Combinations and the 100 Subtraction Combinations.* New York: Longmans, Green and Co., 1928.

54. Knight, F. B. and Setzafandt, A. O. H. "Transfer within a Narrow Mental Function." *Elementary School Journal,* 24 : 780-787, June, 1924.

55. Kramer, Grace A. *The Effect of Certain Factors in the Verbal Arithmetic Problem upon Children's Success in the Solution.* Johns Hopkins University Studies in Education, No. 20. Baltimore: Johns Hopkins Press, 1933.

56. Lazar, May. *Diagnostic and Remedial Work in Arithmetic Fundamentals for Intermediate Grades.* New York: Board of Education, 1928.

57. Lazerte, M. E. *The Development of Problem Solving Ability in Arithmetic.* Toronto: Clarke, Irwin and Company, Ltd., 1933.

58. MacGregor, Gregor. *Achievement Tests in the Primary School.* Publications of the Scottish Council for Research in Education, VI. London: University of London Press, Ltd., 1934.

59. McConnell, T. Raymond. *Discovery vs. Authoritative Identification in the Learning of Children.* Studies in Psychology of Learning, II. Studies in Education, Vol. 9, No. 5, pp. 11-62. Iowa City: University of Iowa Press, 1934.

60. McLaughlin, Katherine. "Number Ability of Pre-school Children." *Childhood Education,* 11 : 348-353, May, 1935.

61. Metter, Harry L. "Trends in the Emphasis on Various Topics since 1860." *Elementary School Journal,* 34 : 767-775, June, 1934.

62. Monroe, Walter S. *How Pupils Solve Problems in Arithmetic.* Bureau of Educational Research Bulletin, Vol. 26, No. 23. Urbana, Illinois: University of Illinois, 1929.

63. Monroe, Walter S. and Engelhart, M. D. "The Effectiveness of

Systematic Instruction in Reading Verbal Problems in Arithmetic." *Elementary School Journal,* 33 : 377-381, January, 1933.

64. Morton, R. L. "An Analysis of Pupils' Errors in Fractions." *Journal of Educational Research,* 9 : 117-125, February, 1924.
65. Morton, R. L. "Pupils' Errors in Solving Arithmetic Problems." *Educational Research Bulletin* (Ohio State University), 4 : 155-158, April, 1925.
66. Morfitt, Margaret D. K. "Comparison of Individual Concrete Methods and Class Methods in the Teaching of Arithmetic." *British Journal of Educational Psychology,* 7 : 196-203, June, 1937, Part II.
67. Neulen, Leon Nelson. *Problem Solving in Arithmetic.* Contributions to Education, No. 483. New York: Bureau of Publications, Teachers College, Columbia University, 1931.
68. Norem, G. M. and Knight, F. B. "The Learning of the One Hundred Multiplication Combinations." *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Education,* Part II, pp. 551-588. Bloomington, Illinois: Public School Publishing Company, 1930.
69. Olander, Herbert T. "Transfer of Learning in Simple Addition and Subtraction." *Elementary School Journal,* 31 : 358-369 and 427-437, January and February, 1931.
70. Olander, Herbert T. and Sharp, E. Preston. "Long Division versus Short Division." *Journal of Educational Research,* 26 : 6-11, September, 1932.
71. Overman, James R. *An Experimental Study of Certain Factors Affecting Transfer of Training in Arithmetic.* Baltimore: Warwick and York, 1931.
72. Polkinghorne, Ada R. "Young Children and Fractions." *Childhood Education,* 11 : 354-358, May, 1935.
73. Pressey, L. C. and Elam, M. K. "The Fundamental Vocabulary of Elementary School Arithmetic." *Elementary School Journal,* 33 : 46-50, September, 1932.
74. Reid, Florence E. "Incidental Number Situations in First Grade." *Journal of Educational Research,* 30 : 36-43, September, 1936.
75. Repp, Austin C. "Mixed versus Isolated Drill Organization." *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Edu-*

cation, Part II, pp. 535-549. Bloomington, Illinois: Public School Publishing Company, 1930.

76. Robinson, Arthur E. *The Professional Education of Elementary Teachers in the Field of Arithmetic.* Contributions to Education, No. 672. New York: Bureau of Publications, Teachers College, Columbia University, 1936.
77. Russell, Ned M. "Arithmetical Concepts of Children." *Journal of Educational Research*, 24 : 647-663, May, 1936.
78. Scarf, R. C. "Special Disability in Reading and Arithmetic." *Twentieth Annual Conference on Educational Measurements.* Bulletin of the School of Education, Indiana University, Vol. X, No. 1, pp. 70-84. Bloomington, Indiana: Bureau of Cooperative Research, Indiana University School of Education, 1933.
79. Sister Mary Immaculata. "Permanence of Improvement and the Distribution of Learning in Addition and Subtraction." *Educational Research Bulletin.* Catholic University of America, Vol. 5, Nos. 9 and 10, 1939.
80. Stone, Cliff W. "Standardized Reasoning Tests in Arithmetic and How To Use Them." Contributions to Education, No. 38. Bureau of Publications, Teachers College, Columbia University, 1916.
81. Stretch, Lorena B. "The Value and Limitations of Drill in Arithmetic." *Childhood Education*, 11 : 413-416, June, 1935.
82. *Studies in Arithmetic.* Vol. 1. Publications of the Scottish Council for Research in Education, XIII. London: University of London Press, Ltd., 1939.
83. Terry, Paul W. *How Numerals Are Read: An Experimental Study of the Reading of Isolated Numerals and Numerals in Arithmetic Problems.* Supplementary Educational Monographs, No. 18. Chicago: Department of Education, University of Chicago, 1922.
84. Thiele, C. L. *The Contribution of Generalization to the Learning of the Addition Facts.* New York Bureau of Publications, Teachers College, Columbia University, 1938.
85. Thiele, C. L. "The Mathematical Viewpoint Applied to the Teaching of Elementary School Arithmetic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics*, pp. 212-232. New York: Bureau of Publications, Teachers College, Columbia University, 1935.
86. Thorndike, E. L. "The Psychology of Drill in Arithmetic: The

Amount of Practice." *Journal of Educational Psychology,* 12 : 183-194, April, 1921.

87. Wahlstrom, E. L. "Computational Arithmetic of Social Experiences of Third Grade Children." *Journal of Educational Research,* 31 : 124-129, October, 1936.
88. Washburne, C. W. "When Should We Teach Arithmetic?" *Elementary School Journal,* 28 : 659-665, May, 1928.
89. Washburne, C. W. "The Work of the Committee of Seven on Grade Placement in Arithmetic." *Child Development and the Curriculum. Thirty-eighth Yearbook of the National Society for the Study of Education,* Part I, pp. 299-324. Bloomington, Illinois: Public School Publishing Company, 1939.
90. Wetherington, Julia. "Grade Placement of the Content of Arithmetic." *Elementary School Journal,* 37 : 41-46, September, 1936.
91. Wheat, H. G. *The Relative Merits of Conventional and Imaginative Types of Problems in Arithmetic.* Contributions to Education, No. 359. Bureau of Publications, Teachers College, Columbia University, 1929.
92. White, Helen M. "Does Experience in the Situation Involved Affect the Solving of a Problm?" *Education,* 54 : 451-455, April, 1934.
93. Wilson, Guy M. "For 100% Subtraction What Method? A New Approach." *Journal of Educational Research,* 27 : 503-508, March, 1934.
94. Wilson, Guy M. *What Arithmetic Shall We Teach?* Boston: Houghton Mifflin Company, 1926.
95. Wilson, Guy M. and Dalrymple, Charles O. "Useful Fractions." *Journal of Educational Research,* 30 : 341-347, January, 1937.
96. Winch, W. H. "'Equal Additions' versus 'Decomposition' in Teaching Subtraction: An Experimental Research." *Journal of Experimental Pedagogy,* 5 : 207-220 and 261-270, June 5 and December 6, 1920.
97. Woody, Clifford. "Achievement in Counting by Children in Primary Grades." *Childhood Education,* 7 : 339-345, March, 1931.
98. Woody, Clifford. "The Arithmetical Backgrounds of Young Children." *Journal of Educational Research,* 24 : 188-201, October, 1931.
99. Woody, Clifford. "Knowledge of Arithmetic Possessed by Young

Children," *Bulletin of the School of Education* (Indiana University), 6 : 50-85, July, 1930.

100. Woody, Clifford. *Nature and Amount of Arithmetic in Types of Reading Material for Elementary Schools.* Ann Arbor: Bureau of Educational Reference and Research, University of Michigan, 1932.

Chapter XV

ONE HUNDRED SELECTED REFERENCES

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EACH day as a part of the daily work of a class, the wise teacher sees to it that the goal is set for the next day. To accord with this plan, the following list of publications has been compiled for the convenience of those teachers in service and in training who desire to extend still further their knowledge of arithmetic and to improve their ability to teach this subject.

The list is not complete. Many other writings could have been listed. Under each heading into which the bibliography has been subdivided the studies which have been included are for the most part the more recent publications which the writer believes will best supplement the content of this Yearbook.

This bibliography has been classified into nine subdivisions as follows:

I. Purposes of Arithmetic Instruction
II. Methods of Teaching
III. Curriculum Studies
IV. The History of Arithmetic and of Mathematics
V. The Psychology of Arithmetic
VI. Diagnosis and Remedial Work
VII. Yearbooks in Arithmetic
VIII. Research Summaries
IX. The Preparation of Teachers

I. PURPOSES OF ARITHMETIC INSTRUCTION

1. Brownell, W. A. "Trends in Primary Arithmetic." *Childhood Education*, 13 : 419-421, May, 1937.

2. Brueckner, Leo J. "The Functions of Instruction in Arithmetic." *Journal of the National Education Association,* 20 : 239-241, October, 1931.
3. Buckingham, B. R. "Arithmetic as a Contribution to a Liberal Education." *Elementary School Journal,* 39 : 577-583, April, 1939.
4. Buckingham, B. R. "Significance, Meaning and Insight—These Three." *Mathematics Teacher,* 31 : 24-30, January, 1938.
5. Buswell, G. T. "Deferred Arithmetic." *Mathematics Teacher,* 31 : 195-200, May, 1938.
6. Griffin, F. L. "Mathematics: A Tool Subject or a System of Thought?" *Mathematics Teacher,* 30 : 223-228, May, 1937.
7. Grimes, Ruby M. "Why Mathematics?" *School Science and Mathematics,* 36 : 426-437, April, 1936.
8. Jacob, Nina. "Personality Adjustment and Development Through Number." *Childhood Education,* 13 : 258-261, February, 1937.
9. Mossman, E. L. "The Primary Purpose of Training in Mathematics—Not a Kit of Tools but a Way of Thinking." *School Science and Mathematics,* 38 : 992-1002, December, 1938.
10. Potter, M. A. "The Human Side of Mathematics." *Mathematics Teacher,* 29 : 123-128, March, 1936.
11. Robinson, A. E. "Are We Teaching Arithmetic Effectively?" *Mathematics Teacher,* 28 : 215-222, April, 1935.
12. Sister Mary Marguerite. "Aims in Arithmetic." *Catholic Education Review,* 35 : 284-293, May, 1937.
13. Smith, E. R. "Mathematics in the School of the Future." *Teachers College Record,* 30 : 795-807, May, 1929.

II. METHODS OF TEACHING

14. Buckingham, B. R. "Trends and Policies in Arithmetic Curricula." *Mathematics Teacher,* 31 : 330-335, November, 1938.
15. Buckingham, B. R. "When to Begin the Teaching of Arithmetic." *Childhood Education,* 11 : 339-343, May, 1935.
16. Clapp, F. L. "Developmental Versus the Informational Type of Material in Teaching Arithmetic." *Reconstructing Education through Research.* Washington: American Educational Research Association, 1936.
17. Deans, E. "Materials and Methods." *Childhood Education,* 11 : 359-366, May, 1935.
18. Dickey, J. W. "Why Invert the Divisor and Multiply?" *School Science and Mathematics,* 36 : 299-302, March, 1936.

19. Foran, T. G. "Methods of Teaching Problem Solving in Arithmetic." *Catholic Education Review,* 32 : 269-282, May, 1934.
20. Harap, H. and Barrett, U. "Experimenting with Real Situations in Third Grade Arithmetic." *Educational Method,* 16 : 188-192, January, 1937.
21. Hedrick, E. R. "Teaching for Transfer of Training in Mathematics." *Mathematics Teacher,* 30 : 51-55, February, 1937.
22. Hooper, L. and Stratton, B. "Developing Number Concepts with Young Children." *Educational Method,* 16 : 193-198, January, 1936.
23. Hughes, H. F. "Character Training from the Arithmetic Lesson." *Educational Method,* 14 : 93-96, November, 1934.
24. Jablonower, J. "Mathematics in a Program of Progressive Education." *Mathematics Teacher,* 31 : 365-367, December, 1938.
25. James, G. "Number and the Four Fundamental Operations of Arithmetic." *School Science and Mathematics,* 37 : 1025-1028, December, 1937.
26. Ligda, P. "The Systematic Solution of Arithmetic Problems." *School Science and Mathematics,* 28 : 24-33, January, 1928; 28 : 172-180, February, 1928.
27. McMurray, F. M. "What is the Matter with Arithmetic?" *Education,* 54 : 449-451, April, 1934.
28. Mitchell, C. "The Specific Type of Problem in Arithmetic Versus the General Type of Problem." *Elementary School Journal,* 29 : 594-596, April, 1929.
29. Morton, R. L. "Higher Decade Addition in Some Recent Drill Devices." *Journal of Educational Research,* 15 : 104-110, February, 1927.
30. Nygaard, P. H. "Accuracy in Addition." *Mathematics Teacher,* 27 : 152-155, March, 1934.
31. Ramseyer, J. A. "The Mathematics Laboratory—A Device for Vitalizing Mathematics." *Mathematics Teacher,* 28 : 228-233, April, 1935.
32. Russell, M. J. "Student Participation in Arithmetic Studies." *Progressive Education,* 15 : 58-64, January, 1938.
33. Thiele, C. L. "Incidental or an Organized Program of Number Teaching?" *Mathematics Teacher,* 31 : 63-67, February, 1938.
34. Voorhees, Margaretta. "New Methods in Arithmetic." *Progressive Education,* 5 : 125-130, April, May, June, 1928.
35. Washburne, C. W. "One Reason Children Fail in Arithmetic." *Progressive Education,* 9 : 215-223, March, 1932.

36. Washburne, C. W. "Teaching of Linear Measure, Square Measure and Time." *Elementary School Journal,* 39 : 509-523, March, 1939.

37. Wilson, G. M. "The Challenge of One Hundred Per Cent Accuracy in the Fundamentals in Arithmetic." *Educational Method,* 15 : 92-96, November, 1935.

38. Woody, C. L. "A General Educator Looks at Arithmetic Readiness." *Mathematics Teacher,* 30 : 314-321, November, 1937.

III. CURRICULUM STUDIES

39. Benezet, L. P. "Story of an Experiment." *National Education Association Journal,* 24 : 241-244, November, 1935; 25 : 7-8, January, 1936.

40. Bond, E. A. "Recent Changes in Point-of-View Relating to the Teaching of Arithmetic." *Mathematics Teacher,* 30 : 175-179, April, 1937.

41. Bond, E. A. "Significant Digits in Computation with Approximate Numbers." *Mathematics Teacher,* 24 : 208-212, April, 1931.

42. Boyce, G. A. "A New Deal for Mathematics." *Progressive Education,* 10 : 280-282, March, 1933.

43. Breslich, E. R. "Mathematics for Grades Seven, Eight, and Nine." *School Science and Mathematics,* 35 : 526-536, May, 1935.

44. Brogan, W. "Science and Arithmetic in the Elementary School Curriculum." *School Science and Mathematics,* 39 : 149-155, February, 1939.

45. Brownell, W. A. "Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics." *Elementary School Journal,* 38 : 495-507, February, 1938.

46. Brownell, W. A. "Place of Crutches in Instruction." *Elementary School Journal,* 34 : 607-619, April, 1934.

47. Brownell, W. A. "Readiness and the Arithmetic Curriculum." *Elementary School Journal,* 38 : 344-354, January, 1938.

48. Brownell, W. A. "Two Kinds of Learning in Arithmetic." *Journal of Educational Research,* 31 : 656-664, May, 1938.

49. Brueckner, L. J. "Deferred Arithmetic." *Mathematics Teacher,* 31 : 287-292, October, 1938.

50. Buswell, G. T. "Relation of Social Arithmetic to Computational Arithmetic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 74-

84. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

51. Lankford, F. G. "Organization of a Course in General Mathematics for Seventh Grade." *Mathematics Teacher,* 29 : 57-60, February, 1936.

52. MacLatchy, J. H. "Reclaiming the Counter." *Educational Research Bulletin,* 16 : 85-90, April, 1937.

53. Spencer, P. L. "Arithmetic: A Basic Social Study." *California Journal of Elementary Education,* 5 : 144-152, February, 1937.

54. Thiele, C. L. "Mathematical Viewpoint Applied to the Teaching of Elementary School Arithmetic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 251-289. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

55. Upton, C. B. "Making Long Division Automatic." *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics,* pp. 251-289. New York: Bureau of Publications, Teachers College, Columbia University, 1935.

56. Washburne, C. W. "Functional Arithmetic." *Educational Method,* 16 : 167-170, January, 1937.

57. Washburne, C. W. "Reply to Brownell's Critique of the Committee of Seven Experiments." *Elementary School Journal,* 39 : 417-430, February, 1939.

58. Washburne, C. W. "Values, Limitations and Applications of the Findings of the Committee of Seven." *Journal of Educational Research,* 29 : 694-707, May, 1935.

59. Woody, Clifford. "When Shall Systematic Instruction in Arithmetic Begin?" *Educational Method,* 16 : 165-166, January, 1937.

IV. THE HISTORY OF ARITHMETIC AND OF MATHEMATICS

60. Karpinski, L. C. *The History of Arithmetic.* Chicago: Rand McNally and Company, 1925.

61. Sanford, Vera. *A Short History of Mathematics.* Boston: Houghton Mifflin Company, 1930.

62. Smith, D. E. *The History of Mathematics.* Boston: Ginn and Company, 1911.

63. Smith, D. E. and Ginsburg, Jekuthiel. *Numbers and Numerals.* New York: Bureau of Publications, Teachers College, Columbia University, 1937.

V. THE PSYCHOLOGY OF ARITHMETIC

64. Breed, F. S. "Our General Outlook on Arithmetic." *Journal of Educational Research,* 32 : 241-254, December, 1938.

65. Brueckner, L. J. "Development of Ability in Arithmetic." *Child Development and the Curriculum. Thirty-eighth Yearbook of the National Society for the Study of Education,* Part I, pp. 275-298. Public School Publishing Company, 1939.

66. Brueckner, L. J. "Diagnosis in Arithmetic." *Educational Diagnosis. Thirty-fourth Yearbook of the National Society for the Study of Education,* Sec. III, pp. 269-302. Bloomington, Illinois: Public School Publishing Company, 1935.

67. Brueckner, L. J. "Vitalized Arithmetic Curriculum." *Journal of Experimental Education,* 1 : 320-322, June, 1933.

68. Dickey, J. W. "Arithmetic and Gestalt Psychology." *Elementary School Journal,* 39 : 46-53, September, 1938.

69. Dickey, J. W. "Readiness for Arithmetic." *Elementary School Journal,* 40 : 592-598, April, 1940.

70. Foran, T. C. "Pupils' Difficulties in Problem Solving." *Catholic Education Review,* 32 : 349-358, June, 1934.

71. Foran, T. G. "Pupils' Methods of Solving Problems in Arithmetic." *Catholic Education Review,* 32 : 154-164, March, 1934.

72. Judd, C. W. "Arithmetic in the Elementary School." *International Understanding through the Public-School Curriculum. Thirty-sixth Yearbook of the National Society for the Study of Education,* Sec. III, pp. 79-84. Bloomington, Illinois: Public School Publishing Company, 1937.

73. Shouse, J. B. "Difficulties of the Concrete." *School Science and Mathematics,* 37 : 937-945, November, 1937.

74. Snedden, D. S. "Proposed Functional Approaches to Improvements of Problem Work in Arithmetic." *Education,* 54 : 461-464, April, 1934.

75. Spear, J. "Mathematics—To Reason Not Just to Do." *School Science and Mathematics,* 38 : 402-410, April, 1938.

76. Tompkins, J. B. and Stokes, C. N. "Eight-Year-Olds Use Arithmetic." *Childhood Education,* 16 : 319-321, March, 1940.

VI. DIAGNOSIS AND REMEDIAL WORK

77. Brueckner, L. J. "Significant Trends in Research in Diagnosis in Arithmetic." *Journal of Educational Research,* 33 : 460-462, February, 1940.

78. Morrison, J. C. "Can We Adapt Arithmetic Instruction to the Needs of All the Children?" *Mathematics Teacher,* 31 : 373-378, December, 1938.

79. Olander, H. T. "The Need for Diagnostic Testing." *Elementary School Journal,* 33 : 736-745, June, 1933.

80. Washburne, C. W. "Primary Grade Origins of Later Remedial Cases." *Childhood Education,* 16 : 29-33, September, 1939.

81. Williams, C. L. "Diagnosis of Arithmetic Difficulties." *Elementary School Journal,* 37 : 592-600, April, 1937.

VII. YEARBOOKS IN ARITHMETIC

82. Department of Superintendence. *Research in Constructing the Elementary School Curriculum. Third Yearbook,* pp. 35-109. G. M. Wilson, Chairman. Washington: Department of Superintendence of the National Education Association of the United States, 1925.

83. Department of Superintendence. *The Nation at Work on the Public School Curriculum. Fourth Yearbook,* pp. 173-320. Washington: Department of Superintendence of the National Education Association of the United States, 1926.

84. Department of Superintendence. *The Superintendent Surveys Supervision. Eighth Yearbook,* pp. 157-161; 302. Washington: Department of Superintendence of the National Education Association of the United States, 1930.

85. Department of Superintendence. *Five Unifying Factors in American Education. Ninth Yearbook,* pp. 136-143. Washington: Department of Superintendence of the National Education Association of the United States, 1931.

86. National Council of Teachers of Mathematics. *The Teaching of Arithmetic. Tenth Yearbook of the National Council of Teachers of Mathematics.* New York: Bureau of Publications, Teachers College, Columbia University, 1935.

87. National Society for the Study of Education. *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Education.* Bloomington, Illinois: Public School Publishing Company, 1930.

VIII. RESEARCH SUMMARIES

88. Brown, J. C. "A Summary of Some Significant Conclusions Reached by Investigators Relative to Arithmetic." *Elementary School Journal,* 25 : 346-357, January, 1925.

89. Buswell, G. T. Annual Summaries and Selected References in the *Elementary School Journal* after 1925.

90. Buswell, G. T. and Judd, C. H. *Summary of Educational Investigations Relating to Arithmetic.* Chicago: University of Chicago Press, 1925.

91. *Review of Educational Research.* Washington: American Educational Research Association. 1931, 354-360; 1932, 140-143; 1933, 14-30; 1937, 160-162 and 453-463.

92. Washburne, C. W. "Work of the Committee of Seven on Grade Placement in Arithmetic." *Child Development and the Curriculum. Thirty-eighth Yearbook of the National Society for the Study of Education,* Part I, pp. 299-324. Bloomington, Illinois: Public School Publishing Company, 1939.

IX. THE PREPARATION OF TEACHERS

93. Bagley, W. C. "The Teachers Professional Study of Subject Matter in Mathematics." *Mathematics Teacher,* 31 : 273-276, October, 1938.

94. Bond, E. A. *The Professional Treatment of the Subject Matter of Arithmetic, Grades One to Six.* New York: Bureau of Publications, Teachers College, Columbia University, 1934.

95. Buckingham, B. R. "The Training of Teachers of Arithmetic." *Report of the Society's Committee on Arithmetic. Twenty-ninth Yearbook of the National Society for the Study of Education,* Part I, Chap. 6. Bloomington, Illinois: Public School Publishing Company, 1930.

96. Morton, R. L. *Teaching Arithmetic in the Elementary School.* Vol. I, *Primary Grades;* Vol. II, *Intermediate Grades;* Vol. III, *Upper Grades.* New York: Silver Burdett Company, 1939.

97. Roantree, William F. and Taylor, Mary S. *An Arithmetic for Teachers.* New York: The Macmillan Company, 1932.

98. Robinson, A. E. "Training of Elementary School Teachers in the Field of Arithmetic." United States Office of Education, Bulletin 10, pp. 474-477, 1933.

99. Wheat, H. G. *Psychology and Teaching of Arithmetic.* Boston: D. C. Heath and Company, 1937.

100. Wilson, G. M., Stone, M., and Dalrymple, C. O. *Teaching the New Arithmetic.* New York: McGraw-Hill Book Company, 1939.